A Hopf algebra structure on rational functions

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We consider an indefinite inner product on the algebra of rational functions over the complex numbers, and we obtain a coproduct, which is dual of the usual multiplication, that gives a structure of infinitesimal coalgebra on the rational functions. We also obtain a representation of the finite dual of the classic polynomial Hopf algebra as a Hopf algebra of proper rational functions.

1 Introduction

The algebra of linearly recurrent sequences has been widely studied because of its multiple applications in several fields, like Number theory, Combinatorics, and Coding theory. See references [1] and [6]. The most recent papers on the subject use the Hopf algebra approach, which has proved to be very fruitful. See references [2] to [5].

In [3], Joni and Rota considered a divided difference coproduct on the polynomial algebra and called the resulting structure an infinitesimal coalgebra. In the present paper we extend such structure to the algebra of rational functions. We introduce an indefinite inner product on the rational functions. With respect to our inner product the divided difference coproduct is dual of the usual multiplication of rational functions.

The inner product also yields a natural injection of the proper rational functions into the dual vector space of the polynomials. Taking the usual Hopf algebra structure on the polynomials we obtain immediately a Hopf algebra structure on the proper rational functions, which is isomorphic to the finite, or continuous, dual of the polynomial Hopf algebra. This finite dual is usually described as the Hopf algebra of linearly recurrent sequences. See [1], [2], [4], and [5]. We also consider a multiplication on the proper rational functions which corresponds to the Hadamard product of linearly recurrent sequences.

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The main tools for our development are the partial fractions decomposition formula and Leibniz’s rule for Taylor functionals. We also use some properties of divided differences functionals, which we have studied in [8], [9], and [10].

In a previous paper [10] we considered the relationships between linearly recurrent sequences and rational generating functions, using an approach based on divided differences, but not very different from the approach used by Stanley in his book [6, Chapter 4]. The present paper complements [10] and focuses on the Hopf algebra aspects. The main references for the basic results on Hopf algebras that we use here are [3], [4], [5], and [7].

This work was motivated by some of the material in Prof. Rota’s lecture notes in Hopf algebras and Combinatorics.

2 Rational functions and partial fractions

We denote by \( \mathcal{P} \) the complex vector space of all polynomials in one indeterminate. For \( n \geq 0 \) the subspace of \( \mathcal{P} \) of all polynomials whose degree is at most \( n \) is denoted by \( \mathcal{P}_n \). The complex vector space of rational functions is denoted by \( \mathbb{Q} \). We say that an element \( f = p/u \) of \( \mathbb{Q} \) is defined at a complex number \( a \) if \( u(a) \neq 0 \).

The Taylor functional \((a, k)^*\) is defined on the set of rational functions \( f \) that are defined at \( a \) as follows

\[
(a, k)^* f = \frac{D^k f(a)}{k!}, \quad a \in \mathbb{C}, \ k \in \mathbb{N},
\]  

(2.1)

where \( D \) denotes the usual differentiation operator. Let us note that

\[
(a, k)^* x^m = \binom{m}{k} a^{m-k}, \quad m, k \in \mathbb{N},
\]  

(2.2)

and

\[
(a, k)^*(x-b)^{-1-m} = (-1)^k \binom{m+k}{k} (a-b)^{-1-m-k}, \quad a \neq b, m, k \in \mathbb{N}.
\]  

(2.3)

Leibniz’s rule for differentiation gives us

\[
(a, n)^*(fg) = \sum_{k=0}^{n} (a, k)^* f \ (a, n-k)^* g,
\]  

(2.4)

for any rational functions \( f \) and \( g \) that are defined at \( a \). If \( g \) is a rational function defined at \( a \) and \( f(x) = (x - a)^m g(x) \), where \( m \geq 0 \), then, by (2.4) we have \((a, n)^* f = 0 \) for \( 0 \leq n \leq m - 1 \). Let us note that

\[
(a, k)^*(x - a)^m = \delta_{k,m}, \quad k, m \in \mathbb{N}.
\]  

(2.5)
Let \( r \geq 0 \), let \( a_0, a_1, \ldots, a_r \) be distinct complex numbers and \( m_0, m_1, \ldots, m_r \) positive integers. Let

\[
u(x) = \prod_{i=0}^{r} (x - a_i)^{m_i}
\]

and \( n + 1 = \sum m_i \). Define the index set \( \mathcal{I} \) by

\[
\mathcal{I} = \{(i, j) : 0 \leq i \leq r, \ 0 \leq j \leq m_i - 1\}
\]

and for each \((i, j)\) in \( \mathcal{I} \) let

\[
q_{i,j}(x) = \frac{\nu(x)}{(x - a_i)^{m_i - j}}.
\]

Note that \( q_{i,j} \) is a polynomial whose degree is at most \( n \). If \( j > 0 \) then \( a_i \) is a root of \( q_{i,j} \) with multiplicity \( j \). Note also that \( a_i \) is not a root of \( q_{i,0} \).

We define the linear functionals \( L_{i,j} \) by

\[
L_{i,j}p = (a_i,j)^* \frac{p}{q_{i,0}}, \quad p \in \mathcal{P}, \ (i,j) \in \mathcal{I}.
\]

Since

\[
\frac{q_{k,s}(x)}{q_{i,0}(x)} = \frac{(x - a_i)^{m_i}}{(x - a_k)^{m_k - s}},
\]

we see that \( L_{i,j}q_{k,s} = 0 \) for \( i \neq k \), and also

\[
L_{i,j}q_{i,s} = (a_i,j)^*(x - a_i)^s = \delta_{j,s}.
\]

Therefore

\[
L_{i,j}q_{k,s} = \delta_{(i,j),(k,s)}, \quad (i,j),(k,s) \in \mathcal{I}.
\]

This clearly implies that the set \( \{q_{i,j} : (i,j) \in \mathcal{I}\} \) is linearly independent and thus it is a basis of the subspace \( \mathcal{P}_n \). Therefore we have

\[
p(x) = \sum_{(i,j) \in \mathcal{I}} L_{i,j}p \ q_{i,j}(x), \quad p \in \mathcal{P}_n.
\]

Dividing the above equation by \( \nu(x) \) we obtain the partial fractions decomposition formula

\[
\frac{p(x)}{\nu(x)} = \sum_{(i,j) \in \mathcal{I}} \frac{L_{i,j}p}{(x - a_i)^{m_i - j}}, \quad p \in \mathcal{P}_n.
\]

We define \( \mathcal{R} \) as the complex vector space of all rational functions of the form \( p/\nu \), where \( \nu \) is monic of positive degree, the degree of \( p \) is strictly less than the degree of \( \nu \), and \( p \) and \( \nu \) have no common factors. The elements of \( \mathcal{R} \) are called proper rational functions. The partial fractions formula says that every element of \( \mathcal{R} \) can be written in a unique way as a finite linear combination of functions
of the form \((x - a)^{-1-k}\), where \(a\) is a complex number and \(k\) is a nonnegative integer. We will use the notation

\[(a, k) = (a, k)(x) = \frac{1}{(x - a)^{1+k}}, \quad a \in \mathbb{C}, \; k \in \mathbb{N}.\]  

(2.12)

The partial fractions formula (2.11) gives us the multiplication formula

\[(a, k)(b, m) = \sum_{j=0}^{k} (a, j)^*(b, m) \cdot (a, k - j) + \sum_{i=0}^{m} (b, i)^*(a, k) \cdot (b, m - i), \quad a \neq b.\]  

(2.13)

We also have \((a, k)(a, m) = (a, k + m + 1)\).

By the division algorithm for polynomials every rational function can be written in a unique way as the sum of a polynomial and a proper rational function. Therefore \(Q = P \oplus R\) as complex vector spaces.

3 An inner product on the rational functions

Let us define

\[\langle (a, k), (b, m) \rangle = \begin{cases} 
0, & \text{if } a = b, \\
(a, k)^*(b, m), & \text{if } a \neq b, 
\end{cases} \quad k, m \in \mathbb{N}.\]  

(3.1)

Note that this is just an extension of the Taylor functional \((a, k)^*\). An equivalent statement is

\[\langle (a, k), (b, m) \rangle = \text{Residue at } a \text{ of } (a, k)(b, m).\]

The above definition and (2.3) show that

\[\langle (b, m), (a, k) \rangle = -\langle (a, k), (b, m) \rangle.\]  

(3.2)

Extending (3.1) by linearity we obtain an indefinite skew-symmetric inner product on \(R\). We can extend the inner product to all of \(Q\) as follows. Define first

\[\langle (a, k), p \rangle = (a, k)^*p, \quad p \in P.\]  

(3.3)

Thus \(\langle f, p \rangle\) is well defined for \(f\) in \(R\) and \(p\) in \(P\), and we can now define

\[\langle p + f, q + g \rangle = \langle f, q \rangle - \langle g, p \rangle + \langle f, g \rangle, \quad p, q \in P, \; f, g \in R.\]  

(3.4)

Note that \(\langle p, q \rangle = 0\) for any polynomials \(p\) and \(q\). Note also that the inner product on \(Q\) is skew-symmetric.

A simple computation using (2.13), or the interpretation of the inner product in terms of residues, gives us

\[\langle (a, n), (a, k)(b, m) \rangle = \langle (a, n + k + 1), (b, m) \rangle, \quad a, b \in \mathbb{C}, \; n, k, m \in \mathbb{N}.\]  

(3.5)
Proposition 3.1 Let \( f \) be an element of \( \mathcal{R} \) and let \( p \) and \( q \) be polynomials. Then
\[
\langle pf, q \rangle = \langle f, pq \rangle. \tag{3.6}
\]

Proof: Let \( k, n, \) and \( m \) be nonnegative integers and let \( a \) be a complex number. Then
\[
\langle (a, k), x^n x^m \rangle = (a, k)^* x^{n+m} = \binom{n+m}{k} a^{n+m-k}.
\]
On the other hand, writing \( x^n = (a + x - a)^n \) we get
\[
x^n(a, k)(x) = \sum_{j=0}^n \binom{n}{j} a^{n-j} (x-a)^{j-1},
\]
and, since the inner product of two polynomials is zero, we obtain
\[
\langle x^n(a, k)(x), x^m \rangle = \sum_{j=0}^r \binom{n}{j} a^{n-j} (a, k-j)^* x^{m+j-1},
\]
where \( r = \min \{k, n\} \). The last equality follows from the Chu-Vandermonde convolution formula. Therefore we get
\[
\langle (a, k)(x), x^n x^m \rangle = \langle x^n(a, k)(x), x^m \rangle,
\]
and (3.6) follows by linearity. \( \square \)

Proposition 3.2 Let \( p \) and \( u \) be polynomials such that \( p/u \) is in \( \mathcal{R} \) and let \( f \) be any rational function. Then
\[
\langle p/u, f \rangle = \langle 1/u, pf \rangle = \sum_i \text{Res}_{a_i} \frac{pf}{u}, \tag{3.7}
\]
where the sum runs over the distinct roots \( a_i \) of \( u \).

Proof: Let \( u \) be as in (2.6). Using the inner product we can write (2.11) in the form
\[
p(x) u(x) = \sum_{(i,j) \in I} \langle (a_i, j), p/q_i, 0 \rangle (a_i, m_i - 1 - j)(x). \tag{3.8}
\]
By Leibniz’s rule we have
\[
\langle p/u, f \rangle = \sum_i \langle (a_i, m_i - 1), pf/q_i, 0 \rangle = \langle 1/u, pf \rangle.
\]

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Since
\[
1/q_{i,0} = (x - a_i)^{m_i}/u(x)
\]
we have
\[
\langle (a_i, m_i - 1) pf/q_{i,0} \rangle = \text{Res}_{u} pf_u,
\]
and this gives us the last equality in (3.7).

The linear functional that sends \( f \) to \( \langle 1/u, f \rangle \), for any rational function \( f \) defined on the roots of \( u \), is called the divided differences functional associated to \( u \). We have studied these functionals in [8], [9], and [10].

Let \( u \) be defined by (2.6) and let \( z \) be any number such that \( u(z) \neq 0 \). Taking \( f(x) = 1/(z - x) \) and using (3.7) and (3.8) we get
\[
\langle p(x) u(x), 1/z - x \rangle = \langle 1/u(x), p(x) z - x \rangle = p(z) u(z) .
\] (3.9)

Now take \( p(x) = x^n \), where \( n + 1 \) is the degree of \( u \). Then we have
\[
\frac{z^n}{u(z)} = \langle 1/u(x), \frac{x^n}{z - x} \rangle.
\]

Let us multiply both sides of the above equation by \( z \) and then replace \( z \) by \( 1/y \). We obtain the generating function
\[
\prod_{i=0}^{r} (1 - ya_i)^{-m_i} = \langle \frac{1}{u(x)}, \frac{x^n}{1 - xy} \rangle.
\] (3.10)

Applying the Taylor functional \((0, k)^*\) with respect to \( y \) we get
\[
\langle 1/u(x), x^{n+k} \rangle = \sum \prod_{i=0}^{r} \binom{k_i + m_i - 1}{k_i} a_i^{k_i},
\] (3.11)
where the sum runs over the multiindices \((k_0, k_1, \ldots, k_r)\) that satisfy \( \sum k_i = k \). Note that (3.11) shows that the divided differences functional of \( u \) applied to any polynomial gives a polynomial function of the roots of \( u \).

Let \( f \) be a rational function. We define the difference quotient
\[
f[x, y] = \frac{f(x) - f(y)}{x - y}.
\]

Let us note that \( f[x, y] \) is defined for all \( x, y \) outside the set of poles of \( f \) and \( f[x, x] = f'(x) \).

The identities
\[
\frac{x^{n+1} - y^{n+1}}{x - y} = \sum_{k=0}^{n} x^k y^{n-k}, \quad n \geq 0,
\] (3.12)
and

\[(a,n)[x,y] = -\sum_{k=0}^{n} (a,k)(x) (a,n-k)(y), \quad n \geq 0, \quad (3.13)\]

which follows from (3.12), show that the difference quotient of any rational function \(f\) is of the form

\[f[x,y] = \sum_{k} f_k(x)\tilde{f}_k(y), \quad (3.14)\]

where the sum is finite and the \(f_k\) and \(\tilde{f}_k\) are rational functions whose poles are contained in the set of poles of \(f\).

Since \(\langle (z,0), f \rangle = f(z)\) for any complex number \(z\) that is not a pole of the rational function \(f\), the partial fractions decomposition

\[\frac{1}{(t-x)(t-y)} = \frac{1}{x-y} \left\{ \frac{1}{t-x} - \frac{1}{t-y} \right\} \quad (3.15)\]

yields

\[f[x,y] = \left\langle \frac{1}{(t-x)(t-y)}, f(t) \right\rangle, \quad f \in \mathbb{Q}, \quad (3.16)\]

for any complex numbers \(x\) and \(y\) for which \(f\) is well defined.

In view of (3.14) we can consider iterated inner products of the form

\[\langle \langle f[x,y], g(x) \rangle, h(y) \rangle = \sum_{k} \langle f_k, g \rangle \langle \tilde{f}_k, h \rangle, \quad f, g, h \in \mathbb{Q}.\]

**Proposition 3.3** Let \(f, g\), and \(h\) be rational functions such that \(f\) and \(gh\) have no common poles. Then

\[\langle f, gh \rangle = \langle \langle -f[x,y], g(x) \rangle, h(y) \rangle. \quad (3.17)\]

**Proof:** If \(f\) is of the form \((a,n)\) then (3.17) follows from (3.13) and Leibniz’s rule. If \(f\) is a polynomial then (3.17) is obtained from

\[\langle (a,k)(b,m), x^{s+1} \rangle = \sum_{j=0}^{s} \langle (a,k), x^{j} \rangle \langle (b,m), x^{s-j} \rangle,\]

which is a particular case of (3.11). From these cases we can see that (3.17) holds in general. \(\blacksquare\)

Taking \(f(t) = 1/(t-z)\) in Proposition 3.3 we get

\[g(z)h(z) = \langle f, gh \rangle = \left\langle \left\langle 1, \frac{1}{(x-z)(y-z)}, g(x) \right\rangle, h(y) \right\rangle, \quad (3.18)\]

which relates multiplication with iterated inner products.
The map that sends $f$ to $-f[x,y]$ can be used to define a coproduct $\Gamma$ on $Q$. In terms of the basic elements of $Q$, $\Gamma$ is defined by $\Gamma 1 = 0$,

$$\Gamma x^{n+1} = - \sum_{k=0}^{n} x^k \otimes x^{n-k}, \quad n \in \mathbb{N},$$

(3.19)

and

$$\Gamma(a, n) = \sum_{k=0}^{n} (a, k) \otimes (a, n - k), \quad a \in \mathbb{C}, \quad n \in \mathbb{N}.$$  

(3.20)

From (3.19) we see that there is no augmentation for $\Gamma$. It is easy to verify that $\Gamma$ is a cocommutative, coassociative coproduct on $Q$ and, by Proposition 3.3 it is dual of the usual multiplication of rational functions.

Since

$$(fg)[x, y] = f(x)g[x, y] + f[x, y]g(y),$$

(3.21)

we see that $\Gamma$ is not an algebra map. It behaves like a derivation. The coalgebra $Q$ with the coproduct $\Gamma$ is an example of an infinitesimal coalgebra. See [3].

4 The duality of $\mathcal{P}$ and $\mathcal{R}$

The inner product on $Q$ gives us a natural linear map from $\mathcal{R}$ into the dual vector space of $\mathcal{P}$. If $f$ is an element of $\mathcal{R}$ then $f^*$ is the functional given by

$$f^* q = \langle f, q \rangle, \quad q \in \mathcal{P}.$$  

(4.1)

It is easy to see that the map that sends $f$ to $f^*$ is injective. If $f = p/u$ then $f^* q = (1/u, pq) = 0$ if and only if $pq$ is a multiple of $u$. Since $p$ and $u$ have no common factors we see that the kernel of $f^*$ is the ideal of $\mathcal{P}$ generated by $u$.

We consider now the polynomial algebra $\mathcal{P}$ with the usual Hopf algebra structure given by the usual multiplication, the coproduct $\Delta$ defined by

$$\Delta x^n = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}, \quad n \in \mathbb{N},$$

(4.2)

the augmentation, which is evaluation at zero, that is, the Taylor functional $(0, 0)^*$, and the antipode $S$, defined by $Sx^n = (-x)^n$ for $n \geq 0$.

The coproduct $\Delta$ induces by duality a product $\ast$ on $\mathcal{R}$, determined by

$$\langle f \ast g, p \rangle = \langle f \otimes g, \Delta p \rangle, \quad f, g \in \mathcal{R}, \quad p \in \mathcal{P}.$$  

In terms of the basic elements the product $\ast$ is given by

$$\langle (a, k) \ast (b, m), x^n \rangle = \sum_{j=0}^{n} \binom{n}{j} \langle (a, k), x^j \rangle \langle (b, m), x^{n-j} \rangle$$  

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\[
(m + k) \binom{n}{m + k} (a + b)^{n-m-k}.
\]
Therefore
\[
(a, k) \star (b, m) = \binom{k + m}{k} (a + b, k + m), \quad a, b \in \mathbb{C}, \; k, m \in \mathbb{N}. \tag{4.3}
\]
The unit for \(\star\) is the element \((0, 0)(x) = x^{-1}\).

The restriction of \(\Gamma\) to \(R\) is clearly a coproduct on \(R\). A simple computation using the Chu-Vandermonde convolution formula shows that \(\Gamma\) is an algebra map with respect to the operation \(\star\).

The augmentation on \(R\) is defined by
\[
\epsilon(a, k) = \delta_{k,0}, \quad k \in \mathbb{N}.
\]
The antipode \(S^*\) in \(R\) is the dual of \(S\) and it is given by
\[
S^*(a, k) = (-1)^k (-a, k), \quad a \in \mathbb{C}, \; k \in \mathbb{N}. \tag{4.4}
\]
Therefore \(R\) with the product \(\star\) and the coproduct \(\Gamma\) is a Hopf algebra, and it is the Hopf algebra dual, or finite dual, of \(\mathcal{P}\). See [2], [4], and [5].

We can identify \(R\) with the algebra of linearly recursive sequences as follows. If \(f = p/u\) is in \(R\) we define the sequence
\[
a_k = \langle p/u, x^k \rangle = \langle 1/u(x), x^k p(x) \rangle, \quad k \geq 0. \tag{4.5}
\]
Let \(E\) denote the shift operator, defined by \(Ea_k = a_{k+1}\) for \(k \geq 0\). By Prop. 3.1 we have
\[
E^j a_k = \left\langle \frac{p(x)}{u(x)}, x^j x^k \right\rangle = \left\langle x^j p(x), x^k \right\rangle, \quad j, k \in \mathbb{N}.
\]
Therefore
\[
u(E)a_k = \left\langle \frac{u(x)p(x)}{u(x)}, x^k \right\rangle = \langle p(x), x^k \rangle = 0, \quad k \geq 0,
\]
and thus the sequence \(a_k\) is linearly recurrent. See [10]. The product \(\star\) corresponds to the Hurwitz product of linearly recurrent sequences. See [2].

The group-like elements of \(R\) are the functions \((a, 0)(x) = 1/(x - a)\), for \(a\) in \(\mathbb{C}\). The only primitive basic element of \(R\) is \((0, 1)\), which corresponds to the linearly recurrent sequence
\[
a_k = \delta_{k,1}, \quad k \geq 0.
\]
The product \(\star\) can be expressed in terms of iterated inner products as follows
\[
g \star f(t) = \left\langle g(y), \left\langle f(x), \frac{1}{t - x - y} \right\rangle \right\rangle, \quad f, g \in R. \tag{4.6}
\]
This is analogous to (3.18) and its proof is a trivial computation.
5  The Hadamard product in \( \mathcal{R} \)

We consider now another bialgebra structure on \( \mathcal{P} \) and the corresponding dual structure on \( \mathcal{R} \). Let us consider \( \mathcal{P} \) with the usual multiplication and the coproduct \( \Phi \) defined by

\[
\Phi x^n = x^n \otimes x^n, \quad n \in \mathbb{N}.
\] (5.1)

The counit \( \epsilon \) is given by \( \epsilon x^n = 1 \) for all \( n \) in \( \mathbb{N} \).

The coproduct \( \Phi \) induces a product \( \bullet \) in \( \mathcal{R} \), called the Hadamard product, in the usual way

\[
\langle f \bullet g, p \rangle = \langle f \otimes g, \Phi p \rangle, \quad f, g \in \mathcal{R}, \quad p \in \mathcal{P}.
\] (5.2)

In terms of the basic elements of \( \mathcal{R} \) the Hadamard product is given by

\[
\langle (a, k) \bullet (b, m), x^n \rangle = \langle (a, k), x^n \rangle \langle (b, m), x^n \rangle, \quad n \geq 0.
\] (5.3)

In order to find an explicit expression for \( (a, k) \bullet (b, m) \) as a linear combination of basic elements we proceed as in (4.6) and we get

\[
(a, k) \bullet (b, m)(t) = \left\langle (a, k)(y), \left\langle (b, m)(x), \frac{1}{t - xy} \right\rangle \right\rangle.
\] (5.4)

A simple computation yields

\[
(a, k) \bullet (b, m) = \sum_{j=0}^{r} \binom{k + m - j}{k - j, m - j, j} a^{m-j} b^{k-j} (ab, k + m - j),
\] (5.5)

where \( r = \min\{k, m\} \).

A direct computation shows that the expression for the product given in (5.5) satisfies (5.3). It is clear that \((1, 0)\) is the unit for the Hadamard product.

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