Use of Models in the Teaching of Linear Algebra [1]

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Abstract: We report results on an approach to teaching linear algebra using models. In particular we are interested in analyzing the use of two theories of mathematics education, namely, Models and Modeling and APOS in the design of a teaching sequence that starts with the proposal of a "real life" decision making problem to the students. We briefly illustrate the possibilities of this methodology through the analysis and description of our classroom experience on a problem related to traffic flow that elicits the use of a system of linear equations and different parameterizations of this system to answer questions on traffic control. We describe cycles of students' work on the problem and discuss the advantages of this approach in terms of students' learning and the possibilities of extending it to other problems and linear algebra concepts.

1 Introduction

The ability of introducing new mathematical concepts through the use of modeling situations in the classroom has received considerable attention in the past few years ([1], [2], [3], [4], [5], [6]). In particular, the Models and Modeling perspective ([7], [8], [9]) has become a popular tool to analyze mathematical thinking when teaching mathematics to elementary and middle school students. Results of these studies suggest that students are able to develop important mathematical concepts when working with appropiately designed "real life"' problems, and through them their motivation for the subject is increased. Less work however has been done in the case of undergraduate mathematics courses ([10], [11], [12], [13], [14]). We are particularly interested in the use of this perspective at undergraduate level to teach linear algebra. Linear Algebra has been recognized as an important subject for a variety of disciplines, and thus has become a compulsory subject in many syllabi. It has also been recognized that Linear Algebra is a difficult subject for most of the students. Carlson et. al ([15], [16]) have done some research regarding the main obstacles faced by students when learning concepts and tools of linear algebra. Their work suggests it is desirable to use problems that go beyond simple exercises, especially if they come from other subject areas. This approach will enrich and motivate a significant learning experience. Sierpinska et. al ([16], [17]) studied the teaching of linear algebra with emphasis on the theoretical and practical dicotomical approach when teaching it and when thinking about it. Hence, for example, the concept of a set of linear equations change meaning depending on the type of thinking approach used, and so do the answers to the problems set. Other authors, as Dubinsky ([18]) stress the need to focus on the abstract nature of the Linear

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Algebra concepts and the importance of not avoiding it when teaching it to help students develop the necessary constructions required by advanced Linear Algebra concepts.

Some research has been developed on the use of APOS theory in designing and implementing activities to teach the main concepts involved in understanding and solving systems of linear equations ([19], [20]). These studies' results indicate that students who have already taken a course on Linear Algebra still show difficulties in formulating and solving systems of linear equations related to real life problems and that students face many difficulties when trying to relate equations to their graphical representation and with the graphical representation of the solution space.

Part of the aim of analyzing the use of modeling problems is the design of activities that promote significant development of mathematical reasoning in a meaningful situation or more realistic setting. These activities have the potential of providing valuable insight for researchers and curriculum developers. Specifically, these activities should give the teacher the opportunity to observe and analyze subtle aspects of each students' mathematical development and a clearer view on the student's process of reasoning, allowing the teacher to observe how students verify and justify their mathematical model, as opposed to just the observation of the failure or success at producing an expected answer. Since the construction of abstract concepts is known to be a difficult process, we consider that the use of modeling activities by themselves can provide the setting for students to use their knowledge and to confront new conceptual needs. These needs can then be addressed in the teaching process by introducing (concept-construction) activities that would help the students make the necessary constructions to learn the abstract concepts of Linear Algebra.

The aim of this research project was to study the possibility of introducing important concepts in a Linear Algebra course through the use of mathematical modeling and the design of activities for students based on a mathematics education theory, and to analyze the results of such an approach in terms of the work developed by the students and in terms of their learning. So, the particular research questions we posed were:

- Is it possible to introduce students in a Linear Algebra course to important mathematical concepts through the use of mathematical modeling?
- Is it possible to design teaching strategies based on mathematics education theories such as APOS and Models and Modeling that can help students in their learning of the main concepts introduced in a first university level course?
- What aspects of the mathematical knowledge of students can be accessed and developed by reflecting upon them and by relating them to new concepts through the use of models and activities designed through the use of a mathematics education theory?

In the following section we first describe the main ideas of the two mathematics education theories used in this experience, Models and Modeling and APOS theory, and we discuss the teaching experience in some detail. Later we present the problem used in the teaching experience and the cycles of students' work that were observed while they worked on the solution of the questions of the problem. We discuss students' difficulties throughout the whole process and the results obtained in terms of students' strategies to solve the problem and in terms of their learning. We conclude by signaling some of the opportunities that this approach opens for the teaching of Linear Algebra as well as some of the difficulties involved when this approach is used in the class.

2 Methodology

As described earlier, two different theories were used in the design of the teaching approach used in this study: APOS theory and Models and Modeling perspective.

Action-Process-Object-Schema (APOS) theory was built on Piaget's work and constructivist ideas ([21], [22]). It intends to model the way students learn advanced mathematical topics to be able to design teaching sequences that can prove effective in students learning, and to analyze the knowledge that students display when solving a particular activity at a particular moment of time. In APOS Theory the mechanism that makes the construction of mathematical knowledge possible is reflective abstraction. Under this framework, an *action conception* of a concept or topic is a transformation of a mathematical object where a subject acts according to an explicit algorithm or procedure, or by means of the use of memorized facts, which can be thought of as externally driven. As a person reflects on his or her actions she is able to take control over them, that is, to interiorize them. Once interiorization has occurred, the transformation of the mathematical object is considered a *process conception*, it is an internal transformation of an action. Processes may be transformed through reversal or coordination with other processes. When a person reflects on actions applied to a particular process and becomes aware of the process as a totality, it can then be considered that a process conception has been encapsulated into an *object conception*. A *mathematical schema* is considered as a collection of action, process and object conceptions, and other previously constructed schemata, which are coordinated and synthesized to form mathematical structures utilized in problem situations ([23], [24]). These schemata (or schemas) evolve as new relations between new and previous action, process, and object conceptions and other schemata are constructed and reconstructed. This structure can be thematized and thus become a mathematical object for the learner. Thematization of a schema describes another way to construct a mathematical object ([24]). The application of APOS theory to describe particular constructions by students requires researchers to develop a genetic decomposition - a description of specific mental constructions a person may make in the process of understanding mathematical concepts and their relationships. A genetic decomposition for a mathematical concept or a topic is not unique, it is a general model about how such concept may be constructed; different researchers can develop diverse genetic decompositions of how students in general construct that particular concept, but, once one is proposed, in order to be used in the design of teaching materials, it needs to be supported by research data from students.

The Models and Modeling approach is a useful theoretical framework for developing model-eliciting activities to help students develop ideas in a meaningful realistic context ([7], [8]). The modeling perspective focuses on the development of conceptual tools which are useful in decision making. Researchers working on this perspective have developed criteria that the problems to be posed to the students must satisfy in order to be successfully applied in the classroom to contribute to the learning process of students. The Models and Modeling perspective's main idea consists in introducing realistic complex situations where students engage in mathematical thinking and complex products and conceptual tools are generated to accomplish the intended goal. These products are constructed during cycles of work and reflection and can be, in each cycle, self-evaluated by students.

Under Lesh's models and modeling approach ([7]) a candidate problem should follow six principles to qualify for such analysis as a model-eliciting activity:

1) *Reality Principle* where the context that motivates the problem is sufficiently realistic

to get the students motivated and have enough mathematical elements so that the modeling activity is not a trivial one.

2) *Model Construction Principle* where the problem setting is rich enough to need mathematical concepts in the development of a model. The need for transforming real situations of the problem to a mathematical language that can be analyzed in this case with linear algebra concepts and techniques.

3) *Self-evaluation Principle* so that students are able to verify their progress and check if their proposed models work according to the real behavior of the situation been modeled. The evaluation should indicate students where the model needs modifying.

4) *Construct Documentation Principle* so that student are able to record their thought process, writing the assumptions and model in algebraic terms. This also allows the teacher to verify progress and evaluate the development in the students train of though, suggesting possible additional activities or new concepts to improve the model.

5) *Construct Generalization Principle* so that the models developed could be generalized to other situations or problems. The model developed should become in it self a new mathematical object which students could apply to other problems and serve as an new analysis tool.

6) *Simplicity Principle* so that the problems is not too complicated to permit analysis by the students, or in need of too much additional information for it to lead to a simple model.

As Dubinksy points out ([22]), the use of models and theories to study mathematics education phenomena can: 1) support prediction, 2) have explanatory power, 3) be applicable to a broad range of phenomena, 4) help organize one's thinking about complex, interrelated phenomena, 5) serve as a tool for analyzing data, and 6) provide a language for communication of ideas about learning that go beyond superficial descriptions. Using ideas from both theories it might be possible to design activities for the classroom where students face rich context problems to work on and which lead them to develop mathematical ideas which can be taken as a starting point in sessions where more controlled activities based on a genetic decomposition are introduced. When this is done, these last activities also respond to students' conceptual needs which arose within the modeling process.

As a first step in this study, the genetic decomposition designed in a previous work by Trigueros et. al ([19]) was used. This genetic decomposition is described by the authors in the following terms and can be represented by the Figure 1, that follows the description:

The schemas an individual must bring to the study of systems of equations are set, function, equality and vector space. This means that understanding of systems of linear equations in the context of Linear Algebra requires that the individual should have constructed coordinations between the actions, processes, objects and other schema that are considered in the construction of each of them. Equation and function objects are coordinated into a function that verifies if a given tuple is a solution of a given equation. This process is encapsulated so that it becomes possible to consider the set of all possible solutions for a given equation.

The equation, set and solution schemas are coordinated to construct a process that takes the intersection of the solution sets of two or more equations in a system. This process is then encapsulated so that it becomes possible to compare two systems in terms of their solution sets, to study their properties and to interpret the systems geometrically when possible.

Schemas for equality and equations are coordinated to construct a process that transforms an equation into an equivalent one. This process is coordinated with the schema for systems to construct a process to find an equivalent system of equations, and a process to determine the solution set from this equivalent form. This process of finding the solution set of a system of equations is encapsulated into an object, and then it is possible to study its properties and to relate it to its geometric interpretation.

Also, the process of constructing an augmented matrix and considering it as the representation of a system is encapsulated so that it becomes possible to perform row operations on the matrix to be able to find the solution set from the reduced form and to compare solution sets associated with different augmented matrices.

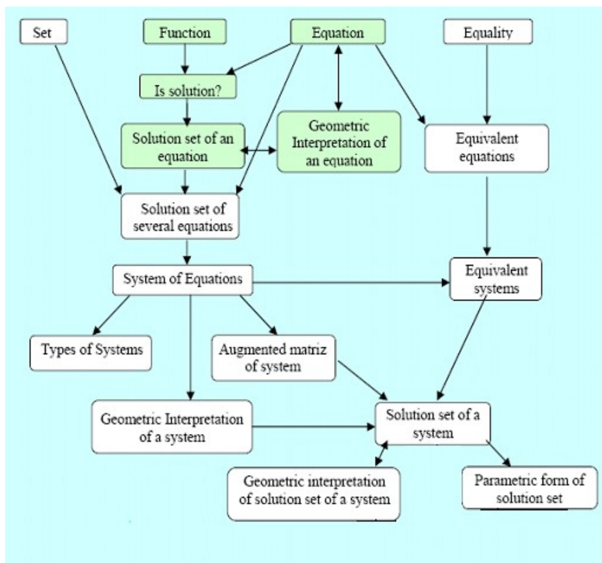


Figure 1: Genetic Descomposition.

Given a particular problem it is our experience that students encounter great difficulties in identifying the variables and the problem conditions that might enable them in setting the linear equations necessary to describe a system of simultaneous equations to model the problem. After a discussion session with a group of researchers, we therefore chose a modeling problem which we considered would allow this to become evident, to identify where the difficulties lie and to promote modeling cycles where students could use their previous or newly constructed knowledge and face new conceptual needs, until the goal of the activity was reached. We also wanted that in the process of exploring different parameterizations, students would find graphical representations for the region of possible parameter values. Exposing our students to these different parameterizations would help them identify an adequate one with which to answer specific questions for the modeling problem. The realistic setting of the problem should motivate this analysis by the students, and serve as a guide for teachers to set activities based on the genetic decomposition. The chosen problem was on traffic flow and will be presented in the next section. It was considered that it satisfied all the requirements posed by the Models and Modeling perspective.

Students in four undergraduate courses on Linear Algebra (Business and Social Sciences, Engineering, and Economics majors), taught by four different teachers, were presented with the problem the first day of term. They worked on the solution of the problem through six class periods of two hours. In each period the session was broken so that students worked on small groups of three students for a while and then in whole class discussion where they could present their advances on the problem and where other students and the teacher could ask questions. In each session all the students' work was collected and classroom discussion was audio registered. After each session the teachers and the researchers had meetings where they analyzed and discussed students' work and designed the conceptual construction activities to be used in the following session and those to be given as homework. Work on these activities was also collected and analyzed by the researchers. Results of the analysis were always negotiated between the researchers for validation.

The analysis of the evolution of schema for linear systems of equations and their solutions and of the interaction between students and with the teacher in relation to the research questions posed above is the focus of this particular study.

3 Modeling Traffic Flow

The context selected for this modeling experience was traffic control in a city. The specific problem posed to the students is described here.

The following diagram (Figure 2) represents a street plan in the busiest first two blocks in the financial district of a city. The traffic control center has installed electronic sensors that count the amount of vehicles passing through specific points in the city. The arrows represent the direction of each street and the numbers the amount of vehicles per hour that pass through that point as accounted by the electronic sensors. At each crossing point there are roundabouts that direct traffic and allow for a continuous flow of traffic through the entire system. Cars are not allowed to park on the streets.

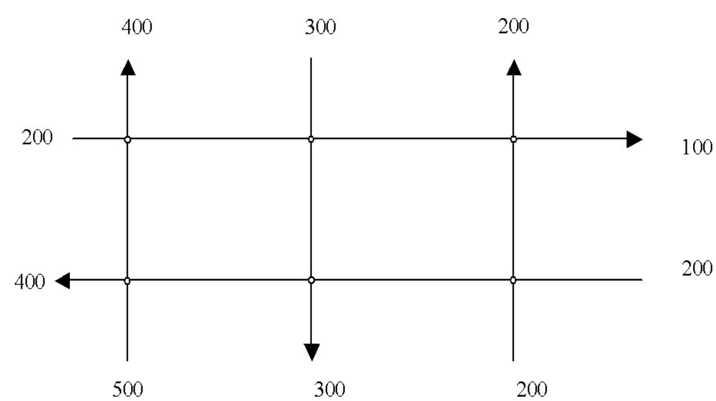


Figure 2: Street Plan and Flow of Traffic

The traffic flow should be allowed to follow its usual course at the sensor points. However the Traffic Control Center is interesting in analysis possible traffic diversion policies. These policies are necessary when road works take place or other special traffic disruption events occur. The students are presented with the following specific questions:

1.- If we were able to set minimum quantities of cars to circulate in a particular road (stretch between roundabout), what would this amount be for each stretch to maintain the normal flow of traffic in the system? Is it possible to close off one of the roads? If so, which ones can or cannot be closed?

2.- The Traffic Control Centre can divert traffic by closing off some of the roads. This is done by installing diverting signs at the begging of each road. How many of such signs are needed ? Is it possible to use them at the beginning of any road? Is there a particular selection of road signs that would make it easier to perform the flow evaluation?

3.- Is your model well adapted to consider a restriction of no more than 200 cars each hour in a particular street? How would you modify it?

3.1 The solution cycles

In the four groups studied four general cycles could be identified in students' work on the solution of the problem, namely:

1) Selecting and relating variables.

2) Student manipulation of the set of linear equations.

3) Matrix representation and its algebraic manipulation.

4) Answering specific questions and the graphical representation of the solution space.

Each cycle was characterized by the type of work that students were doing, and in each of them some specific difficulties were detected. In what follows we briefly describe results obtained in each of them.

3.2 Defining variables and the system

During the first phase students were working in small groups exploring, trying to make sense of the problem and finding possible ways to answer the questions. Students addressed the problem by analyzing the flow of cars in terms of the given numbers. In the beginning variables were not used. Students answered some of the questions by looking at the given numbers in the diagram. They performed arithmetical operations to decide if a street could or could not be closed. Their discussion was centered on ways of defining what "a street" was, and in other considerations such as if a car could disappear because it parks inside of a house or building, and whether it was possible to have two-way streets. Students got easily involved with the problem, and showed that they were motivated and enthusiastic about the activity. The teachers in each classroom visited the small groups and observed what students were doing. They answered students' specific questions to help them understand the problem but did not give any hint about its possible solution.

Figure 3 shows one such example where students try answering specific questions by looking at plausible flows or scenarios. In this particular one students suggested closing the road they later named x_3 and specifying how to direct traffic as not to interrupt the required inbound and outbound flows. Sometimes students were inadvertively making assumptions about where the individual cars wanted to go, as in their point e) where they suggests "the 200 cars coming in from the bottom do a U turn[3] and go out as the 200 cars on top".

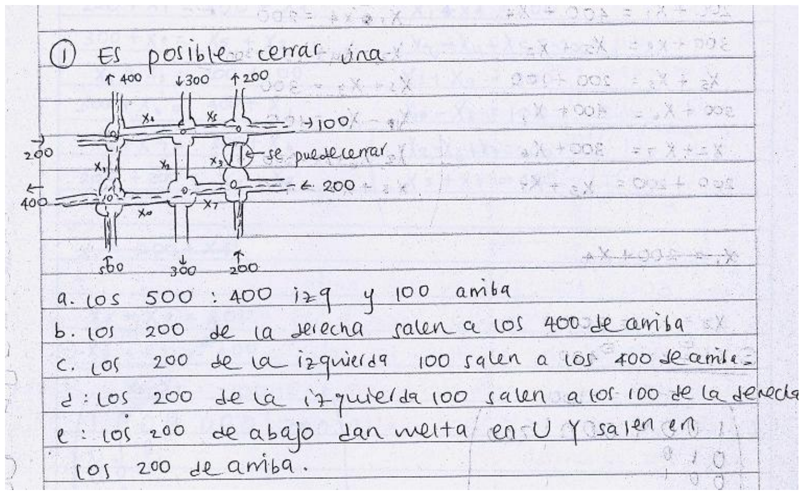


Figure 3: Trying out flows

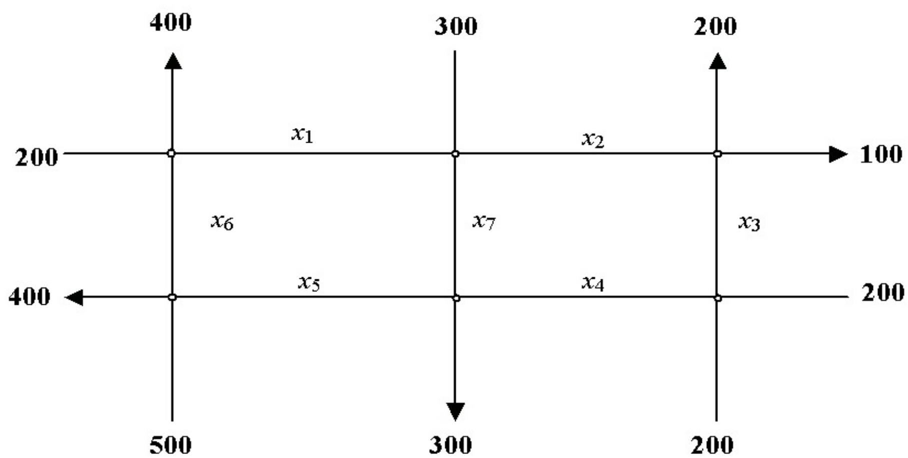
During the beginning of this first cycle, it was observed that students find it hard to identify what the variables are. They tend to omit the key word "number of" in their answers. They would make comments such as "The variables are: the cars, the streets, the roundabouts, etc.".

Teachers found that it was an illuminating exercise to allow students to suggest what the variables were. This allowed the teachers to identify deficiencies in the students' concept of a variable, and helped them to guide the group into a plausible choice of variables. It was useful to allow for the exchange of opinions during whole group discussions that followed a half an hour period of small groups work. In all the four classes there was at least one group of students who designated something on the diagram as a variable. During whole class discussion, where different groups presented their approach to the problem and were questioned both by other students and the teacher, it was found that understanding the flow of cars in the roundabouts as zero (balancing out) was characteristic that permits the setting of linear equations to represent the system. Also, the usefulness of presenting very long answers involving detailed arguments and complicated explanations that included many arithmetical operations was questioned. After discussion, when groups returned to work, all of the groups decided to use variables and to pose the problem in a mathematical form, they all used the term "street" as representing a stretch of inner-road (those where we are not counting the flow of cars) between the roundabouts. The selection of the variables was different for each group, and most of them added a new hypothesis to avoid cars from "disappearing". These results show that the problem satisfied the reality principle from Models and Modeling perspective.

In this problem, there are several valid ways of choosing the variables and then pose a system of equations. One such way is to define x_i as the amount of cars circulating in a

[3]By "U turn" they mean taking street named x_2 and making a right to the topmost rightmost roundabout

particular street, selected by most of the groups. One should notice that to maintain the normal flow of traffic it is necessary that the amount of cars coming into a roundabout be the same going out of it; students did not find it difficult to take this into account. One possibility is to name each inner-road as in the following graphical representation that was chosen by one group:



Then we would have six simultaneous equations (one for each roundabout) described by:

$$\begin{aligned} 200 + x_6 &= 400 + x_1 \\ x_1 + 300 &= x_2 + x_7 \\ x_2 + x_3 &= 200 + 100 \\ 200 + 200 &= x_3 + x_4 \\ x_7 + x_4 &= x_5 + 300 \\ 500 + x_5 &= 400 + x_6 \end{aligned}$$

As considering a negative flow of traffic is unrealistic it is also important to specify that $x_i \in Z^+$, which was also easy for the students to understand, although they did not mentioned this explicitly when they presented their model for the traffic flow. Once students got to this point they tried to establish simplified algebraic expressions for the system, but some of them still tried substituting numbers. Some students tried to solve the equations as they started setting them, and many of them made mistakes since they are not used to working with systems with so many unknowns, and ended up with incorrect answers on plausible flows. During whole class discussion students had an opportunity to talk about equivalent systems and non uniqueness of equations. Here is where Lesh principle of self evaluation comes into play, as it is easy to verify if, when substituting the values obtained or proposed, a feasible flow is obtained. That is why setting the three questions proposed for the problem can be considered a good decision; they provided a guide for the student's train of thought.

Students soon realized that it was easier to avoid mistakes in the algebraic manipulations if they expressed the system of linear equations in such a way that all variables are in one side of the equality and the constants on the other side for each equation as in:

$$\begin{array}{rrrrrrrcr}
-x_1 & & & & & +x_6 & & = & 200 \\
x_1 & -x_2 & & & & & -x_7 & = & -300 \\
& x_2 & +x_3 & & & & & = & 300 \\
& & -x_3 & -x_4 & & & & = & -400 \\
& & & x_4 & -x_5 & & +x_7 & = & 300 \\
& & & & x_5 & -x_6 & & = & -100
\end{array}$$

Although writing the system in this way allows for a more orderly and easier manipulation of the different linear equations, not all groups of students were able to solve the system successfully. Figure 4 shows one such example where students found it difficult to manipulate the system to answer the questions successfully.

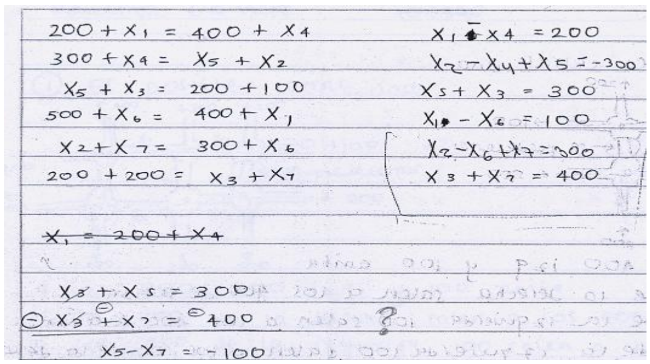


Figure 4: Solving System of equations unsucessfully.

Problems with algebraic manipulation motivated the need for a systematic algorithm to solve system of equations, and for the matrix representation. It was then decided that a set of activities designed on the basis of the genetic decomposition of the system of linear equations should be used in order to give students the opportunity to review what they already knew about systems of linear equations and to develop this knowledge further by performing actions on the system to transform it and by reflecting on those actions in order to interiorize the solution process.

An example of an activity from this set, together with its analysis in terms of the genetic decomposition follows:

Activity

a) Find the solution set of the system:

$$\begin{aligned} -4x_1 + 5x_2 + 9x_3 &= \ -9 \\ 2x_1 - 3x_2 + 2x_3 &= \ \ \ 1 \end{aligned}$$

b) How many solution does the system has? Give some examples of solution vectors.

The purpose of this activity was to give opportunities to students to coordinate the equation, set and solution schema by means of reflecting on the actions needed to find the solution of the system, of performing actions on the equations to transform them in equivalent ones, and determining the solution of the problem. The solution set of the system consists of an infinite number of solutions, so students will not find a specific number for each variable as a solution. It is expected that they will reflect on the fact that the solution is given in functional form, and by coordinating the solution set with the function schema, they will be able to think of the role of free variables and their meaning regarding the number of solutions to the system. Asking students for specific examples of solutions intends to make them perform the action of substituting a value into the independent variable of each function and obtain the other values. Once students have one or more specific vectors that are solution to the system, they can verify this by substituting each of them into the equations and by verifying the identity.

As in the traffic flow problem students showed a tendency to loose some equations in the solution process, some of the actions they were asked to do were such that they realized that they always maintained the six equations of the system. This first set of activities was left as homework for the students.

The following session was started by using a new set of activities based on the genetic decomposition. This set had the purpose of helping the students realize that the numbers that really matter for algebraic manipulation are the coefficients of each variable in each linear equation, and of using this fact to naturally introduce them to the concept of a matrix or matrix representation of a set of linear equalities as an action. The following activity shows an example of this set:

Activity

Given the system of equations:

$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= 11 \\ x_1 + x_2 + x_3 &= 1 \\ 4x_1 + 2x_2 + 3x_3 &= 5 \end{aligned}$$

a) Find the solution set by following a similar procedure to the one described in the previous activity. As you did in that activity, write down each of the systems you obtain while you do the transformation on the left side of the page.

b) On the right hand side of the same page, write down only the numbers corresponding to the coefficients of the unknowns and to the constant term conserving the same layout of the equations. What do you observe? Would it be possible to find the solution set of the system using only this array of numbers? How would you do it?

Part a) of the activity has the goal of helping students reflect on the actions they do to the system to find the solution and to help them realize the fact that these transformations do not reduce the number of equations of the system. Writing the system in this way allows for a more orderly and easier manipulation of the different linear equations. In fact students will note at this point that the numbers that really matter for algebraic manipulation are the coefficients of each variable in each linear equation. This could lead naturally to the concept of a matrix or matrix representation of a set of linear equalities.

In part b) of the activity students do the action of constructing an array of numbers obtained from each transformation of the system of equations with the goal of focusing their attention on the fact that the transformations change the coefficients and the independent term but not the variables. This can serve as preparation for the introduction of the augmented matrix and the actions needed in the gaussian elimination procedure which are new concepts for the students. It can also promote the coordination between the information given by each array and that given by the corresponding system of equations.

3.3 Matrix representation

During the following session, students returned to the solution of their model and most of them were able to solve it. They did not have many difficulties using what they had learnt to represent the system using a matrix. For example some students defined matrix A as follows:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix},$$

During the discussion with the class it was suggested by the teacher that the decision variables may be expressed as a vector $x^T = (x_1, x_2, \ldots, x_6)$, and the right hand side constants by vector: $b^T = (200, -300, 300, -400, 300, -100)$, and that the system could be represented using the matrix form:

$$Ax = b.$$

The teachers also took this opportunity to point out that this matrix form represents the layout of the streets, as follows: each row identifies a specific roundabout and each column a specific inner street. This is in fact an adjacency matrix. So that, for example, the first row represents the first roundabout and a minus sign in the first column indicates that the first inner street goes out of the first roundabout and a positive sign in the sixth row means that inner road number six enters the roundabout. The goal of this discussion was to elicit the use of adjacency matrix as an object to represent the road layout.

The idea of clarifying the relationship of the notation introduced to the meaning of the problem intended to help students to use it as a tool to represent more complex layouts. For example a case where there are more than two roads coming into a roundabout, or more complex arrangements where the road layout is not as regular or square-like as presented in the example. We can observe here that the problem satisfies the principle of generalization, also, the possibility to relate systems to their matrix representation and both with the streets' layout can help students to construct an object conception of a system of equations and of its matrix representation.

Another interesting behavior that was pointed out by one of the teachers during discussion with the whole class is that each column has one positive one (+1) and one negative one (-1), which indicates that the inner road starts at a specific roundabout (corresponding to the row where the negative sign is) and ends at another specific roundabout (where the positive sign is). This teacher used this opportunity to introduce the concept of a unimodular matrix and the integrality theorem regarding solutions to a system with such matrix (that may be proven with Cramer's rule). Specifically as matrix A is a unimodular matrix and b is integer, we are able to find integer solutions to the system (an integer number of cars flowing on each road), which is convenient (and meets the expected result of the reality of the model).

Returning to the system of linear equations in the matrix form Ax=b, a natural concept to introduce is that of the Gaussian elimination method to obtain an echelon matrix, this again consists of performing actions on the rows of the matrix to transform it into equivalent matrices. These actions may be interiorized into a process of row operation. This was done by introducing a new set of activities designed using the genetic decomposition. After the set of activities was completed, students applied the method to their model and ended up with a system as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} -200 \\ 100 \\ 200 \\ 200 \\ -100 \\ 0 \end{pmatrix}$$

A group of students was not able to transform the final matrix obtained into the corresponding system of equations. Some groups of students substituted but did not know what to do with the free values, others substituted one possible value for the free variables and obtained a particular solution to the problem. There were other groups, however, that realized that there were many possibilities for the solution of the problem. These groups were able to explore solutions to the whole set of questions posed at the beginning. They were able to compare different solutions and they tried to find the best solution possible. These students constructed the idea of solution set as an object.

The teachers discussed with students that there were two degrees of freedom. This intended to help them reflect upon the process of solution and the solution as an object. Also, different groups' models were compared so that students could notice that a different choice of free variables will yield different parameterizations of the solutions to the system and that some will facilitate the analysis needed to answer questions while others will not be as easy to analyze. This comparison could help students to encapsulate the notion of solution set as an object.

3.4 Parameterization

The teachers invited the students to write their solutions using parameters. For example, some groups selected as parameters x_6, and x_7 by choosing to name $x_6 = t_1$, and $x_t = t_2$ as free variables, and expressed the set of solutions to the traffic flow problem as follows:

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} -200 \\ 100 \\ 200 \\ 200 \\ -100 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

At this point all the groups of students played around with their model substituting different values for their parameters and tried to answer the questions posed at the beginning when the problem was introduced. One of the first things they noticed is that as $\underline{x} \in Z^+$ it is necessary that $t_1 \geq 200$ (so that $x_1 \geq 0$), and were able to express in words the meaning of this restriction: "the street corresponding to x_2 should at least have a flow of 200 cars". They also compared their different parameterizations in terms of how easy it was for each of them to answer the questions. Some groups of students also used the parameterization to analyze other characteristics of the feasible solutions. One of the teachers suggested during whole group discussion, on the meaning of those restrictions for the problem, the

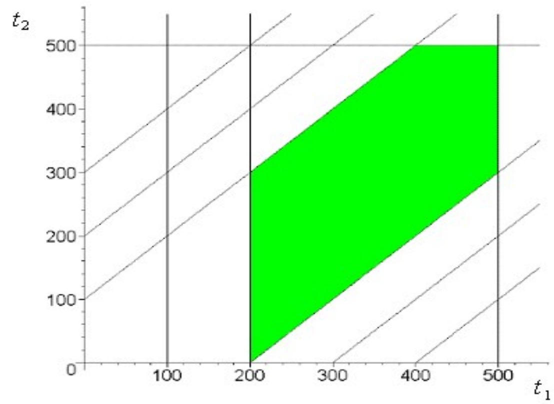


Figure 5: Parameterization with $x_6 = t_1$, and $x_7 = t_2$.

possibility to graph the feasible values of t_1 and t_2, and asked students to do it for their specific parameterization.

The graph in Figure 5 describes a graph for a parameterization. Some students realized that the shaded area represents the feasible region for values of t_1 and t_2 and it was then evident for them that "as $t_1 = 200$ that the flow through road x_6 must be at least of 200 cars, and that as $t_2 = 500$ road x_7 cannot possible support more than 500 cars going through it". This graph was also used by other students to distinguish if given specific values for t_1 and t_2 were a feasible or an infeasible choice. During discussion of work on parameterization the teacher asked questions to help the whole groups to reflect on the parameterization as an object representing the solution set, on the relation between the graphic representation of the parameterization, the solution set and the solution of the questions asked at the beginning. He also asked questions to help students note that with a different choice the analysis might become easier, as follows:

If we choose different parameters, say, $x_1 = s_1$ and $x_2 = s_2$ then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 300 \\ 100 \\ 100 \\ 200 \\ 300 \end{pmatrix} + s_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

Its corresponding graph is shown in Figure 6.

Here it is easier to see that $x_2 \leq 300$, the flow in road x_2 and not exceed 300 cars and hour. We also conclude that we can close off road x_1 and x_2 (making $s_1 = s_2 = 0$), or close road x_3 (making $s_2 = 300$). We also see that as $s_1, s_2 \geq 0$ (to mantain feasibility of $\underline{x} \geq 0$) we cannot possible close off roads x_4, x_5 and x_6, and the only way we could close off road x_7 would be to make $s_1 = 0$ and $s_2 = 300$, which means that we also close road x_1, and x_3 diverting all traffic through the other roads. It is now easier also to answer question 3, as it corresponds to focusing on a smaller region in this graph. One where $s_1 \leq 200$, $s_2 \leq 200$, and from $x_3 = 300 - s_2$: $s_2 \geq 100$. Through this discussion the teacher tried to help the students interiorize the process of coordinating the value ranges and the

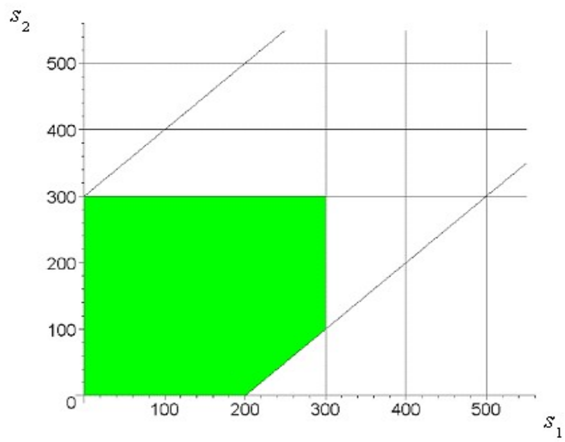


Figure 6: Parameterization with $x_1 = s_1$, and $x_2 = s_2$.

graphical interpretation to answer specific questions. The final activity for students was a set of different open problems that could be modeled using the same ideas as those used in the traffic flow problem, in this way, the traffic problem becomes a useful tool that students can use to compare a variety of different situations and start a process where they will eventually become aware of the mathematical structure relating those apparently diverse problems.

4 Conclusions

A first issue to be investigated in terms of the students' activity consisted in determining if the problem situation presented satisfies the Models and Modeling criteria.

- It was found that students can make sense of the presented situation by using the mathematical schemas they have already constructed, and to reflect on the concepts they know in terms of a new situation, that is, the problem satisfies the reality principle.

- When working with the problem, it was found that students very quickly face a situation where they need to extend and contrast their knowledge about variables, functions, equations and systems of equations. This shows that the problem satisfies the principle of model construction.

- Throughout the work with the problem students were able to go back from the theoretical activities proposed to the model, they used the newly acquired knowledge in their proposed solution and judged by themselves the convenience both of the model they proposed and of the mathematical tools used to work with it. This shows that the problem satisfies the auto-evaluation principle.

- The modeling activities, together with the mathematical activities presented to the students were used to keep track of the different groups thought evolution throughout the whole modeling process. Documentation could also be used to discuss with students and design mathematical activities needed and appropriate feedback on the modeling process. The documentation principle is thus satisfied.

- Students were able to recognize new open problems presented to them as similar to the "streets model", they used their models and the new conceptual tools to work in the new problems. This implies that students could use de "streets model" as a generalization tool.

- Finally, the situation presented was simple enough to enable students to start working of the problem using numerical and algebraic strategies, so the problem satisfies the simplicity principle.

It was also shown that both theoretical frameworks could be used in an integrated way: Rich contextual problems can be introduced for the students to find a suitable mathematical model. Once students have a model they work with it to find answers to the questions posed. From students' work on the problem, teachers and researchers can have discussions both on the conceptual constructions that students show in their work and on the concepts they need to construct to be able to continue their work on the problem. Based on the genetic decomposition teachers and researchers can work on the design of activities that can help students construct new knowledge. These activities ideally should be linked to the modeling situation in terms of the detected student needs. Students may work on them in small groups during class sessions or they can be left as homework. Work on the activities should then be discussed with the whole class, different models compared and new concepts are formalized.

During work with the whole class the activities that were introduced were related to: matrix representation of systems, row operations, geometric representation of systems of linear equations, solution of the system, classification of systems, types of solution of linear system, geometric representation of solution sets, inverse matrices, matrix rank, linear programming, and in some cases other useful mathematical concepts.

The main difficulties faced by students when presented to the problem can be summarized as follows:

- A drive to look for an immediate solution to the problem which leads to the use of numerical calculations, very specific graphical representations or the proposal of a mathematical model not related directly with the problem.

- Difficulty in recognizing some hypothesis that were already stated in the proposition of the problem, and in adding, when needed, additional hypothesis.

- Difficulties in identifying the relevant variables and in using and interpreting parameters in the proposed models.

- Difficulty in finding an appropriate mathematical model for the problem and, for those stated, difficulties in interpreting the model.

- Difficulties with the concepts of function and variation.

The teachers found that it is of fundamental importance to guide the students with questions that help them to reflect on what they already know and focus on strategies that can be fruitful in the solution of the problem. An important question is thus, once the students have found one or several models, how can they be guided so that new concepts emerge and can be related with what they already know?

Work done on this research project shows that it is possible to teach new Linear Algebra concepts to students through the use of rich contextual problems, and that the use of Mathematics Education theories is helpful in the interpretation of students work and of students needs, as well as in the design of concept construction activities that are linked to the modeling process as well.

It is important to stress that it was also possible to use two different Mathematics Education theories to design an interesting real situation which can be worked by students and to help them make the concept constructions intended in the course syllabus in relation to systems of linear equations.

The genetic decomposition is an useful tool to guide teaching decisions and the design of activities with the purpose of making students conceptual schema evolve. That ideas of the models and modeling theory are useful in the design and evaluation of the problems to use. Formalization of new concepts through whole class discussion after each cycle was completed was fundamental to make students aware of what they had achieved through their own work in terms of mathematical accepted knowledge.

The strategy followed through the whole modeling process gave opportunities for students to show what they know and what they are learning. As all students work is documented, this documentation becomes an important tool in the evaluation of students' progress. Students showed a lot of interest throughout the whole process, and worked intensely on the proposed activities.

It is also important to note that the use of this teaching strategy needs a lot of work from the teacher, it is easy to loose track and loose time. Sometimes students prefer to be taught by traditional methods, but it has been well documented through mathematics education literature that there are teaching methods that are more efficient in terms of students learning.

Results of this study show that students learn what they are supposed to learn and they can even do more than what is normally expected from them when given the opportunity. In this experience, the geometrical analysis of the solution space and the parameters came out as a direct result of students' work on the model.

The design of a genetic decomposition is not easy, but there are several already available in the literature that can be used or modified by the teacher. There are also sets of activities designed to help students make the constructions needed in the learning of Linear Algebra ([25]).

This project is undertaken by the authors in collaboration with other researchers[4]. The group has chosen other problems that work in a similar way to the one presented here to introduce other concepts, specifically to do with: distance between vectors, matrix multiplication, eigenvalues and eigenvector, linear independence, among others. Problems that are under analysis at the moment and we hope to present in future papers.

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