Nonzero patterns for nonsingular sign regular matrices*

V. CORTES AND J.M. PEÑA

Departamento de Matemática Aplicada/IUMA, Universidad de Zaragoza, 50009 Zaragoza, Spain
vcortes@unizar.es, jmpena@unizar.es

Abstract. Depending on the signature of nonsingular sign regular matrices, many nonzero entries can be guaranteed. These patterns for their nonzero entries are analyzed.

AMS subject classifications. 15A48, 15A15.

Key words. Sign regular matrices, Nonzero patterns.

1. Introduction

An $n \times n$ matrix is called *sign regular of order* $r \leq n$ if for each $k \leq r$ all its $k \times k$ minors have the same sign or are zero. If $r = n$, the matrix is called *sign regular*. The common sign ε_k may differ for different k, and the sequence $(\varepsilon_1, \ldots, \varepsilon_n)$ is the *signature* of the matrix. These matrices appear in many fields due to their variation diminishing property: see, for instance, [1], [2] (for statistical applications), [6] and [7] (for applications to computer aided geometric design). Recent results and algorithms for sign regular matrices can be found in [3–5]. A very important subclass of the sign regular matrices is formed by the totally nonnegative matrices. A matrix is *totally nonnegative* if all its minors are nonnegative.

It is well known (see, for instance, Corollary 3.8 of [1]) that the principal minors of nonsingular totally nonnegative are nonzero. In particular, their diagonal entries are nonzero. In [8] it was shown that, for other nonsingular sign regular matrices, we can also guarantee that special entries are nonzero, depending on the signature of the matrix. This brief paper continues this analysis, finding many more signatures of nonsingular sign regular matrices with special patterns for nonzero entries. In addition, it clarifies some aspects of [8].

* Research partially supported by the Spanish Research Grant MTM2006-03388, Gobierno de Aragón and Fondo Social Europeo.

2. Main results

Let us start by introducing some basic notations. Given $k, n \in \mathbf{Z}$, $1 \leq k \leq n$, $Q_{k,n}$ will denote the set of all increasing sequences of k natural numbers less than or equal to n. If $\alpha \in Q_{k,n}$, the *complement* $\alpha' \in Q_{n-k,n}$ is the increasingly rearranged sequence $\{1, 2, \ldots, n\} \setminus \alpha$. Let A be a real $m \times n$ matrix. For $k \leq m$, $l \leq n$, and for any $\alpha \in Q_{k,m}$ and $\beta \in Q_{l,n}$, we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of A containing rows numbered by α and columns numbered by β. The principal submatrices will be written in the form $A[\alpha] := A[\alpha|\alpha]$. A matrix is called nonnegative (nonpositive) if it has nonnegative (nonpositive) entries.

For an $n \times n$ matrix C with $C[\gamma]$ invertible, the *Schur complement of* $C[\gamma]$ *in* C, denoted by $C/C[\gamma]$, is defined as

$$C/C[\gamma] = C[\gamma'] - C[\gamma'|\gamma](C[\gamma])^{-1}C[\gamma|\gamma']$$

Then

$$\det C[\gamma] = \frac{\det C}{\det(C/C[\gamma])}. \tag{1}$$

For nonsingular matrices A, Gaussian elimination consists of a succession of $n-1$ steps resulting in a sequence of matrices as follows:

$$A = A^{(1)} \to A^{(2)} \to \cdots \to A^{(n)} = U,$$

where U is an upper triangular matrix. At the end of step $t-1$, the matrix $A^{(t)} = (a_{ij}^{(t)})_{1 \leq i,j \leq n}$ will have been constructed, having zeros its main diagonal in its $t-1$ first columns. To obtain $A^{(t+1)}$ from $A^{(t)}$ we produce zeros in column t below the *pivot element* $a_{tt}^{(t)}$ by subtracting multiples of row t from the rows beneath it. Since we have assumed that A is nonsingular, it is well-known (use (1)) that, if no row exchanges are needed, one has, for $i \geq t$, $j \geq t$,

$$a_{ij}^{(t)} = \frac{\det A[1, 2, \ldots, t-1, i|1, 2, \ldots, t-1, j]}{\det A[1, 2, \ldots, t-1]}. \tag{2}$$

Finally, given a positive real number r, we denote by $\lfloor r \rfloor$ the greatest integer less than or equal to r.

Remark 1. *Let A be a nonsingular sign regular matrix with signature $(\varepsilon_1, \ldots, \varepsilon_n)$ and let $P := (\delta_{n-i+1,j})_{1 \leq i,j \leq n}$ be the backward identity matrix of order n, obtained by reversing the order of the rows of the identity matrix I of orden n. Clearly P is a nonsingular sign regular matrix with signature $\varepsilon_k = (-1)^{\lfloor \frac{k}{2} \rfloor}$ for all $k = 1, \ldots, n$. By Theorem 3.1 of [1], PA is a nonsingular sign regular matrix with signature $(\varepsilon_1, -\varepsilon_2, \ldots, (-1)^{\lfloor \frac{n}{2} \rfloor}\varepsilon_n)$. So, PAP is also a nonsingular sign-regular matrix with the same signature of A.*

Suppose that the signature sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of an $n \times n$ nonsingular sign regular matrix A satisfies one of the following four cases:

(a) $\varepsilon_i = +1 \quad \forall i = 1, \ldots, n$,

(b) $\varepsilon_i = (-1)^i \quad \forall i = 1, \dots, n$,
(c) $\varepsilon_i = (-1)^{\lfloor \frac{i}{2} \rfloor} \quad \forall i = 1, \dots, n$,
(d) $\varepsilon_i = (-1)^{\lfloor \frac{i}{2} \rfloor + i} \quad \forall i = 1, \dots, n$.

These cases (a), (b), (c) and (d) appear when A, $-A$, PA or $-PA$, respectively, is totally nonnegative. In these cases we can only guarantee that the $n \times n$ nonsingular sign regular matrix has n nonzero entries. In fact, the maximal number of zero entries can be achieved, as shown by the matrices I (the identity matrix of order n), $-I$, P or $-P$, respectively. Depending on the signature ε, we can guarantee more nonzero entries, as the following results show. We start by the following auxiliary result contained in Theorem 2.1 of [8], which holds for any nonsingular sign regular matrix of order 3.

Lemma 2. *Let $A = (a_{ij})_{1 \le i,j \le n}$ be a nonsingular sign regular matrix of order 3 with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$.*

(i) If $\varepsilon_1 = \varepsilon_3$ and $\varepsilon_2 = -1$, then $a_{ij} \neq 0$ whenever $(i,j) \notin \{(1,1),\ (n,n)\}$.
(ii) If $\varepsilon_1 \neq \varepsilon_3$ and $\varepsilon_2 = +1$, then $a_{ij} \neq 0$ whenever $(i,j) \notin \{(1,n),\ (n,1)\}$.
(iii) If $\varepsilon_1 \neq \varepsilon_3$ and $\varepsilon_2 = -1$, then $a_{n-i+1,i} \neq 0$ for all $i = 1, \dots, n$.
(iv) If $\varepsilon_1 = \varepsilon_3$ and $\varepsilon_2 = +1$, then $a_{ii} \neq 0$ for all $i = 1, \dots, n$.

Observe that the comment of the first paragraph of page 95 of [8] corresponds to the signatures (a)–(d) presented above instead of the signatures (2.2)–(2.5) of [8]. The same confusion happened in Theorem 2.2 of [8]. In fact, the following result has the same proof as that of Theorem 2.2 of [8], and we include it for the sake of completeness and clarity. The cases (2.8) and (2.9) of the mentioned Theorem 2.2 of [8] correspond now to the cases (i) and (ii) of Theorem 4, which need a new proof.

Theorem 3. *Let $A = (a_{ij})_{1 \le i,j \le n}$ be a nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ such that (a) – (d) do not hold. Then there exists a positive integer k with $2 < k \le n$ such that one of the following possibilities holds:*

(i) $\quad \varepsilon_i = +1 \quad \forall i < k, \quad \varepsilon_k = -1$,
(ii) $\quad \varepsilon_i = (-1)^i \quad \forall i < k, \quad \varepsilon_k = (-1)^{k-1}$,
(iii) $\quad \varepsilon_i = (-1)^{\lfloor \frac{i}{2} \rfloor} \quad \forall i < k, \quad \varepsilon_k = (-1)^{\lfloor \frac{k}{2} \rfloor + 1}$,
(iv) $\quad \varepsilon_i = (-1)^{\lfloor \frac{i}{2} \rfloor + i} \quad \forall i < k, \quad \varepsilon_k = (-1)^{\lfloor \frac{k}{2} \rfloor + k + 1}$.

If either (i) or (ii) holds, then $a_{ij} \neq 0$ whenever $|i-j| < n-k+2$. If either (iii) or (iv) holds, then $a_{ij} \neq 0$ whenever $k \le |i+j| \le 2n-k+2$.

Proof. The existence of k is obvious. Let us assume that (i) holds and let us prove the result by induction on $k \ge 3$. The case $k = 3$ follows from Lemma 2 (ii) of [8]. Let us assume that the result holds for $k-1$ and let us prove it for $k > 3$.

By Theorem 2.1 (iv) of [8], $a_{11} > 0$, and so we can perform a step of Gaussian elimination to produce zeros in the first column of A below a_{11} and obtaining the matrix $A^{(2)} = (a_{ij}^{(2)})_{1 \le i,j \le n}$. By the formula (2), $A^{(2)}[2, \dots, n]$ is an $(n-1) \times (n-1)$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_2, \dots, \varepsilon_n)$ such that (i) holds replacing k by $k-1$. Applying the induction hypothesis to $A^{(2)}[2, \dots, n]$, we deduce that $a_{ij}^{(2)} \neq 0$ for all $i, j \ge 2$ such that $|i-j| < n-1-(k-1)+2 = n-k+2$. Since the matrix $A^{(2)}$ is nonnegative and we obtain A from $A^{(2)}$ by adding to each row a nonnegative multiple of the first row, the mentioned nonzero entries of $A^{(2)}$ are also nonzero entries of A. Since PAP is also a nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ such that (i) holds we get that entries (i, j) of PAP are nonzero for all $i, j \geq 2$ such that $|i - j| < n - k + 2$. In conclusion, $a_{ij} \neq 0$ whenever $|i - j| < n - k + 2$.

If we assume that A is nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ such that (ii) ((iii) or (iv)) holds, then $-A$ (PA or $-PA$, respectively) is a nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ such that (i) holds and the result follows. ■

The following result presents new nonzero patterns that can appear on a nonsingular sign regular matrix depending on the signature ε.

Theorem 4. *Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. Let k be a positive integer with $2 < k \leq n$ and such that such that one of the following possibilities holds:*

(i) $\varepsilon_1 = +1, \quad \varepsilon_i = -1 \quad \forall i \in \{2, \ldots, k-1\}, \quad \varepsilon_k = +1,$

(ii) $\varepsilon_1 = -1, \quad \varepsilon_i = (-1)^{i+1} \quad \forall i \in \{2, \ldots, k-1\}, \quad \varepsilon_k = (-1)^k,$

(iii) $\varepsilon_1 = +1, \quad \varepsilon_i = (-1)^{\lfloor \frac{i}{2} \rfloor + 1} \quad \forall i \in \{2, \ldots, k-1\}, \quad \varepsilon_k = (-1)^{\lfloor \frac{k}{2} \rfloor},$

(iv) $\varepsilon_1 = -1, \quad \varepsilon_i = (-1)^{\lfloor \frac{i}{2} \rfloor + i + 1} \quad \forall i \in \{2, \ldots, k-1\}, \quad \varepsilon_k = (-1)^{\lfloor \frac{k}{2} \rfloor + k}.$

If either (i) or (ii) holds and $k = 4$, then $a_{n-i+1,i} \neq 0$ for all $i = 1, \ldots, n$. If either (iii) or (iv) holds and $k = 4$, then $a_{ii} \neq 0$ for all $i = 1, \ldots, n$. If either (i) or (ii) holds and $k \neq 4$, then $a_{ij} \neq 0$ whenever $(i, j) \notin \{(r, s) \,|\, s > n - r + 1 \quad \text{and} \quad r, s > n - k + 2\}$ for $j > n - i + 1$, or $(i, j) \notin \{(r, s) \,|\, s < n - r + 1 \quad \text{and} \quad r, s < k - 1\}$ for $j < n - i + 1$. If either (iii) or (iv) holds and $k \neq 4$, then $a_{ij} \neq 0$ whenever $(i, j) \notin \{(r, s) \,|\, s > r \quad \text{and} \quad r < k - 1,\, s > n - k + 2\}$ for $j > i$, or $(i, j) \notin \{(r, s) \,|\, s < r \quad \text{and} \quad r > n - k + 2,\, s < k - 1\}$ for $j < i$.

Proof. Let us assume that (i) holds and let us prove the result by induction on $k \geq 3$. The case $k = 3$ follows from Lemma 2 (i). Now, let us suppose that $k > 3$ and $a_{ij} = 0$ for some i, j. Since A is nonsingular, we could find $t \neq i$ such that $a_{tj} \neq 0$. If $t < i$ then $a_{im} = 0$ for all $m > j$ because otherwise $\det A[t, i|j, m] = a_{tj} a_{im} > 0$, which is a contradiction. Again by the nonsingularity of A, $a_{ih} \neq 0$ for some $h < j$. Then $a_{km} = 0$ for all $k > i$, $m \geq j$ because otherwise $\det A[i, k|h, m] = a_{ih} a_{km} > 0$, which is a contradiction and we have proved that $a_{km} = 0$ for all $k \geq i$ and for all $m \geq j$. Since A is nonsingular $j > n - i + 1$. Analogously, it can be proved that, if $t > i$, then $a_{km} = 0$ for all $k \leq i$, $m \leq j$ and, since A is nonsingular, $j < n - i + 1$. As a consequence, $a_{n-i+1,i} \neq 0$ for all $i = 1, \ldots, n$. So, the result holds for (i) with $k = 4$.

Now, let us assume that (i) holds and $k > 4$. We also suppose that $a_{ij} = 0$ for $j > n - i + 1$ and $j \leq n - k + 2$. Then, by the previous reasoning, $a_{rs} = 0$ $\forall r \geq i$, $s \geq j$. Let us now assume that k is odd (resp., k is even). Since A is nonsingular, there exist $\alpha, \beta \in Q_{k-1,n}$ with $\alpha_{k-1} < i$ and $\beta_1 = n - k + 2$ (resp., $\alpha, \beta \in Q_{k-3,n}$ with $\alpha_{k-3} < i$ and $\beta_1 = n - k + 4$) such that $\det A[\alpha|\beta] \neq 0$. Now, we consider the submatrix $A[\alpha, n|1, \beta]$ of A which has $a_{n1} \neq 0$ and $a_{n\beta_i} = 0$ for all $i = 1, \ldots, k - 1$ (resp., for all $i = 1, \ldots, k - 3$). By expanding $\det A[\alpha, n|1, \beta]$ on its last row, we have

$$\det A[\alpha, n|1, \beta] = a_{n1} \det A[\alpha|\beta]. \tag{3}$$

By (3) and (i), the minors of order k (resp., order $k-2$) of the sign regular matrix A have sign $\varepsilon_k = \varepsilon_1\varepsilon_{k-1} = -1$ (resp., $\varepsilon_{k-2} \neq \varepsilon_1\varepsilon_{k-3} = -1$, so that $\varepsilon_{k-2} = +1$) which is a contradiction with $\varepsilon_k = +1$ (resp. $\varepsilon_{k-2} = -1$) defined in (i).

Now we suppose that $a_{ij} = 0$ for $j > n-i+1$ and $i \leq n-k+2$. Then, by the reasoning of the first paragraph of the proof, $a_{rs} = 0$ for all $r \geq i$, $s \geq j$. Let us assume again that k odd (resp., even. Since A is nonsingular, there exist $\alpha, \beta \in Q_{k-1,n}$ with $\alpha_1 = n-k+2$ and $\beta_{k-1} < j$ (resp., $\alpha, \beta \in Q_{k-3,n}$ with $\alpha_1 = n-k+4$ and $\beta_{k-1} < j$) such that $\det A[\alpha|\beta] \neq 0$. Now, we consider the submatrix $A[1,\alpha|\beta,n]$ of A which has $a_{1n} \neq 0$ and $a_{\alpha_i n} = 0\,\forall i = 1,\ldots,k-1$ (resp., $\forall i = 1,\ldots,k-3$). By expanding $\det A[1,\alpha|\beta,n]$ on its last column, we have

$$\det A[1,\alpha|\beta,n] = a_{1n} \det A[\alpha|\beta]. \tag{4}$$

By (4) and (i), the minors of order k (resp., order $k-2$) of the sign regular matrix A have sign $\varepsilon_k = \varepsilon_1\varepsilon_{k-1} = -1$ (resp., $\varepsilon_{k-2} \neq \varepsilon_1\varepsilon_{k-3} = -1$, so that $\varepsilon_{k-2} = +1$) which is a contradiction with $\varepsilon_k = +1$ (resp., $\varepsilon_{k-2} = -1$) defined in (i). Therefore, we have proved that if $a_{ij} = 0$ with $j > n-i+j$, then $(i,j) \in \{(r,s) | r > n-s+1 \quad \text{and} \quad r,s > n-k+2\}$.

Analogously, we can prove that if $a_{ij} = 0$ with $j < n-i+1$, then $(i,j) \in \{(r,s) | s < n-r+1 \quad \text{and} \quad r,s < k-1\}$. For this purpose, we replace A by the nonsingular sign regular matrix PAP, which has the same signature of A by Remark 1.

Finally, if we suppose that A is a sign regular matrix with signature $(\varepsilon_1,\ldots,\varepsilon_n)$ such that (ii) ((iii) or (iv), respectively) holds, then $-A$ (PA or $-PA$, respectively) is a nonsingular sign regular matrix with signature $(\varepsilon_1,\ldots,\varepsilon_n)$ such that (i) holds and the result follows. ■

References

[1] T. Ando, *Totally positive matrices*, Linear Algebra Appl. 90 (1987), 165–219.

[2] L.D. Brown, I.M. Johnstone and K.B. MacGibbon, *Variation Diminishing transformations: a direct approach to total positivity and its statistical applications*, Journal of the American Statistical Association 76 (1981) 824-832.

[3] V. Cortés and J.M. Peña, *Sign regular matrices and Neville elimination*, Linear Algebra Appl. 421 (2007), 53–62.

[4] V. Cortés and J.M. Peña, *Decompositions of strictly sign regular matrices*, Linear Algebra Appl. 429 (2008), 1071–1081.

[5] V. Cortés and J.M. Peña, *A stable test for strict sign regularity*, Math. Comp. 77 (2008), 2155–2171.

[6] S. Karlin, *Total Positivity*, Stanford University Press, Stanford, 1968.

[7] J.M. Peña (Ed.), *Shape preserving representations in Computer-Aided Geometric Design*, Nova Science Publishers, Commack (New York), 1999.

[8] J. M. Peña, *On nonsingular sign regular matrices*, Linear Algebra Appl. 359 (2003), 91–100.