Perturbations preserving conditioned invariant subspaces

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Abstract

Given the set of vertical pairs of matrices $\mathcal{M} \subset M_{N+n,N}(\mathbb{C})$ keeping the subspace $\mathbb{C}^d \times \{0\} \subset \mathbb{C}^N$ invariant, we obtain the implicit form of a miniversal deformation of a pair belonging to an open dense subset of $\mathcal{M}$. We compute this deformation explicitly when the pair is observable and the subspace $\mathbb{C}^d \times \{0\}$ is marked. Moreover, we obtain the dimension of the orbit, characterize the structurally stable vertical pairs and study the effect of each deformation parameter.

Keywords: vertical pairs of matrices; conditioned invariant subspaces; marked pairs; stratified manifold; miniversal deformation; dimension of the orbits; bifurcation diagrams

AMS subject classifications: 93B07, 93B27, 93C05

1 Introduction

This paper completes the study of versal deformations when square matrices or pairs of matrices are considered, together with invariant subspaces. Versal deformations were introduced by Arnold in [1] (see also [2]) to study the variations of the invariants of a square matrix when its entries are perturbed. Thanks to a natural generalization contained in [24], the same technique has been applied to other cases, such as perturbations of pairs of matrices representing linear systems ([12], [13]).

Invariant subspaces play a key role both in square matrices (see [18]) and linear systems (see [25]), where they are often called "conditioned" invariant subspaces. The differentiable structure of the set of invariant subspaces of a square matrix has been studied in [23] and that of conditioned invariant subspaces of a pair in [16] and [17]. In the context of versal

⋆ Partially supported by DGICYT MTM2007-67812-C02-02.

deformations, invariant subspaces arise in a natural way. For instance, in the Carlson problem (that is, the possible Segre characteristic of a block-triangular nilpotent matrix when diagonal blocks are prescribed), one asks for perturbations of the given matrix preserving a prefixed invariant subspace. Also, in the miniversal deformation of a pair of matrices in [12], the initial controllability subspace is preserved as invariant subspace of the perturbed pairs.

Moreover, in [18] the "interesting class" of the so-called *marked* subspaces, namely, the invariant subspaces having a Jordan basis which can be extended to a Jordan basis of the whole space, is introduced. For instance, in [6] one proves that the "simplest" solutions of the Carlson problem are marked, and any other appears in a neighborhood of the marked ones. This notion was extended to pairs of matrices in [5] and used in [7] for the analogue to the Carlson problem: again the marked solutions cover all the possibilities and are the simplest realizations.

It seems natural to consider the situation when both a matrix and an invariant subspace are involved and both or one of the elements of this couple is perturbed. So, in [10] one describes a versal deformation of a couple composed of an endomorphism and an invariant subspace when both are perturbed. More explicit constructions are possible when only one of the elements is perturbed. Thus, in [15] one obtains the equations of a versal deformation of an invariant subspace with regard to a fixed endomorphism, which are explicitly solved in [8] for marked subspaces. On the other hand, in [9] one studies the perturbation of a matrix preserving an invariant subspace, again mainly when it is marked. In particular, as we have pointed out, this perturbation gives all the solutions of the Carlson problem, and hence explicit realizations can be obtained (see [7]).

Analogously, one can consider the couple formed by a linear system and a conditioned invariant subspace, in particular a marked one. The versal deformation of this subspace (with regard to a fixed pair of matrices) is studied in [11] and [21] whereas here we complete the cycle, tackling the perturbation of the linear system preserving a given conditioned invariant subspace. We have already mentioned that this situation appears in [12]. Again we focus our attention on the marked case, which as above, has interesting properties; for instance "minimal" observable perturbations of a non-observable pair are marked.

We obtain the equations of a miniversal deformation of a pair preserving a given conditioned invariant subspace and solve them explicitly for two particular cases, obtaining "minimal" solutions (that is, without repeated parameters). Firstly, when the preserved conditioned invariant subspace is a supplementary of the unobservable one. Then one obtains just the versal deformation of a pair in [12] which, in this sense, is generalized here. In particular, this family contains marked observable perturbations of the initial pair. In fact, as we have pointed out, some of the "simplest" ones in order to the initial pair becomes observable. They are just the second particular case which we solve explicitly (when the initial pair is nilpotent). Moreover, we remark a nice geometric fact: the minimal miniversal deformation of each one of these second cases, is a subfamily of that obtained for the initial pair; in other words, the miniversal deformation of each perturbed pair is contained in that of the initial pair.

Some applications are derived: computation of the dimension of the orbits, characterization of structurally stable objects, study of bifurcations diagrams... In addition, the orbits and the versal deformations here obtained are compared with those in [12], where the ordinary block similarity is considered: for a nilpotent BK-matrix, the results are the same for both equivalence relations; for observable marked pairs of matrices, the miniversal deformation in [12] is a strict subfamily of that here.

More specifically, we consider the set $\mathcal{M} \subset M_{N+n,N}(\mathbb{C})$ of pairs of matrices having $\mathbb{C}^d(\equiv \mathbb{C}^d \times \{0\} \subset \mathbb{C}^N)$ as a conditioned invariant subspace (2.2). Two pairs of matrices will be called equivalent if they are block-similar and the change of basis in the state space $\mathbb{C}^N$ induces an automorphism in $\mathbb{C}^d$ (2.5). Our aim is to study these equivalence classes and their variations when the pair preserving $\mathbb{C}^d$ as conditioned invariant subspace is perturbed, mainly when the subspace is marked (2.4).

If we fix a basis adapted to $\mathbb{C}^d \subset \mathbb{C}^N \subset \mathbb{C}^{N+n}$, in (3.2) we prove that the elements of $\mathcal{M}$ are those of the form

$$a = \begin{pmatrix} A_1 & A_3 \\ FC_1 & A_2 \\ \hline C_1 & C_2 \end{pmatrix}.$$

Then $\mathcal{M}$ can be differentially stratified by $\operatorname{rank} C_1$ (3.4) and the above equivalence classes are the orbits under the action on it of a suitable group (3.6). This is the starting point to apply Arnold's techniques.

We restrict ourselves to the maximal stratum $\mathcal{M}^*$, which is an open dense set in $\mathcal{M}$ (4.1). Then the equations of a miniversal deformation are obtained in (4.2). We solve them explicitly when $a \in \mathcal{M}^*$ is a BK-matrix (4.3) or it is marked and observable (5.6), in which case a quite simple canonical form is available (5.1). As an application we compute the dimension of the orbits (6.1) and characterize the structurally stable pairs (6.3).

Moreover, a simpler miniversal deformation having a nicer pattern is derived (5.8). In particular, it facilitates the study of the effect of each deformation parameter (section (6.3)). In addition, this simplified pattern allows us to compare the versal deformation obtained here with the one when block similarity is considered (section (6.4)).

The organization of this paper is as follows. In section (2), we summarize some prerequisites concerning conditioned invariant subspaces, marked subspaces... Section (3) is devoted to the study of the differentiable structure of $\mathcal{M}$ as a stratified manifold. In section (4), we obtain the equations of a miniversal deformation of a pair $a \in \mathcal{M}^*$, which are solved in section (5) when $a \in \mathcal{M}^*$ is observable and marked, also obtaining a second miniversal deformation without repeated parameters. Finally, in section (6) some applications are derived.

We use the following notation. We write $M_{p,q}(\mathbb{C})$ the set of complex matrices having p rows and q columns. We denote by $\|A\|$ the usual norm of the matrix A and consider the usual hermitian product $< A, B >= \operatorname{trace}(AB^*)$, where B^* means the conjugate-transpose matrix of B. We denote by B^t the transpose matrix of B. If $p = q$, we simply write $M_p(\mathbb{C})$, and $Gl(p)$ will be the group of non-singular matrices in it.

2 Prerequisites

We will deal with matrices of the form $\begin{pmatrix} A \\ C \end{pmatrix}$, or equivalently pairs of matrices $(C, A) \in M_{n,N}(\mathbb{C}) \times M_N(\mathbb{C})$, which will be simply denoted as a if no confusion is possible. Such pairs can be reduced to the following form:

Theorem 2.1 *[14] Given $(C, A) \in M_{n,N}(\mathbb{C}) \times M_N(\mathbb{C})$, there exist integers $k_1 \geq k_2 \geq \cdots \geq k_r > 0$, $k_1 + \cdots + k_r = N$ called* Brunovsky Kronecker indices *(or simply* BK-indices*) and a convenient basis called* BK-basis *in which the matrix of the pair, hereafter called* BK-canonical form*, is*

$$a_{BK} = \left(\begin{array}{cc} N & 0 \\ 0 & J \\ \hline E & 0 \\ 0 & 0 \end{array}\right),$$

where

(i) $N = diag(N_1, \ldots, N_r)$, with $N_i \in M_{k_i}(\mathbb{C})$, $1 \leq i \leq r$, a lower nilpotent block,

(ii) $E = diag(E_1, \ldots, E_r)$, with $E_i = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} \in M_{1,k_i}(\mathbb{C})$, $1 \leq i \leq r$.

(iii) J is a Jordan matrix.

The block J does not appear if (C, A) is observable *or, equivalently, if (A^t, C^t) is controllable.*

Let us consider a subspace $S \subset \mathbb{C}^N$, $\dim S = d$, and bases adapted to it, that is to say, whose d first vectors form a basis of S. Then we will assume that the matrices A and C are block-partitioned into

$$A = \begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

where $A_1 \in M_d(\mathbb{C})$, $\quad C_1 \in M_{n,d}(\mathbb{C})$.

Definition 2.2 *[18] A subspace $S \subset \mathbb{C}^N$ is* (C, A)-invariant *or* conditioned invariant *if $A(S \cap \operatorname{Ker} C) \subset S$. Equivalently (see 1.8.5 [4]), if there is a basis adapted to S such that* *the pair becomes*

$$\begin{pmatrix} \bar{A} \\ \bar{C} \end{pmatrix} = \left(\begin{array}{cc} A_1 & A_3 \\ 0 & A_2 \\ \hline C_1 & C_2 \end{array}\right),$$

with $A_1 \in M_d(\mathbb{C}), C_1 \in M_{n,d}(\mathbb{C}), d = \dim S$. *Then we identify* $S = \mathbb{C}^d \times \{0\}$, *and one says that* (C_1, A_1) *is the* restriction *to* S *of the given pair* (C, A).

Remark 2.3 *In [20] and [22] an intrinsic geometric definition of the restriction of a pair to a conditioned invariant subspace is presented. One can also consider the "quotient map" defined in* $\{0\} \times \mathbb{C}^{N-d}$ *and prove (see [4]) that it is an endomorphism (having matrix* A_2*) if and only if* $C_2 = 0$.

Generalizing the concept of a marked subspace with regard to an endomorphism, we say that a (C, A)-invariant subspace is marked if there is some BK-basis of the restriction which can be extended to a BK-basis of (C, A):

Definition 2.4 *[5] Let* $S \subset \mathbb{C}^N$ *be a* (C, A)*-invariant subspace.* S *is said to be* (C, A)*-*marked *if there exists an adapted basis to* S *in which the matrix of the pair* (C, A) *has the form*

$$\begin{pmatrix} \bar{A} \\ \bar{C} \end{pmatrix} = \left(\begin{array}{cc} A_1 & A_3 \\ 0 & A_2 \\ \hline C'_1 & C'_3 \\ 0 & C'_2 \end{array}\right),$$

and

(i) (C'_1, A_1) *is a BK-matrix.*
(ii) $(\bar{C}, \bar{A})$ *is a BK-matrix, except for permutations, that is, there exists a permutation matrix* $P \in M_N(\mathbb{C})$ *such that* $(\bar{C}P, P^t\bar{A}P)$ *is a BK-matrix.*

Then we say that $(\bar{C}, \bar{A})$ *is a* marked matrix *(with regard to* S*).*

When conditioned invariant subspaces are involved, the usual block similarity between pairs of matrices is restricted in a natural way:

Definition 2.5 *[21] Given two pairs of matrices* $(C, A), (C', A') \in M_{n,N}(\mathbb{C}) \times M_N(\mathbb{C})$ *having* $S = \mathbb{C}^d \times \{0\} \subset \mathbb{C}^N$ *as a conditioned invariant subspace, we say that they are* S*-equivalent (or simply equivalent if no confusion is possible) if*

(i) The pairs are block-similar, that is, $A' = PAP^{-1} + RCP^{-1}$, $C' = QCP^{-1}$, *where* $P \in Gl(N)$, $Q \in Gl(n)$, $R \in M_{N,n}(\mathbb{C})$.
(ii) The change of basis P *keeps* S *invariant.*

More explicitly,

$$\begin{pmatrix} A' \\ C' \end{pmatrix} = \left(\begin{array}{cc|c} P_1 & P_3 & R_1 \\ 0 & P_2 & R_2 \\ \hline 0 & 0 & Q \end{array}\right) \left(\begin{array}{cc} A_1 & A_3 \\ 0 & A_2 \\ \hline C_1 & C_2 \end{array}\right) \begin{pmatrix} P_1 & P_3 \\ 0 & P_2 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} P_1A_1P_1^{-1} + R_1C_1P_1^{-1} & -(P_1A_1 + P_3FC_1 + R_1C_1)P_1^{-1}P_3P_2^{-1} + (P_1A_3 + P_3A_2 + R_1C_2)P_2^{-1} \\ R_2C_1P_1^{-1} & -(P_2FC_1 + R_2C_1)P_1^{-1}P_3P_2^{-1} + (P_2A_2 + R_2C_2)P_2^{-1} \\ QC_1P_1^{-1} & -QC_1P_1^{-1}P_3P_2^{-1} + QC_2P_2^{-1} \end{pmatrix}$$

Conversely, bearing in mind that

$$\left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & -R_2Q^{-1} \\ \hline 0 & 0 & I \end{array}\right) \left(\begin{array}{cc} A_1' & A_3' \\ A_4' & A_2' \\ \hline C_1' & C_2' \end{array}\right) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \left(\begin{array}{cc} A_1' & \cdots \\ 0 & \cdots \\ \hline C_1' & \cdots \end{array}\right),$$

we have:

Proposition 2.6 *If $S = \mathbb{C}^d \times \{0\}$ is a (C, A)-invariant subspace and (C', A') is obtained from (C, A) as above, then S is also (C', A')-invariant and both restrictions, i.e. (C_1', A_1') and (C_1, A_1), are block-similar. In particular,* $\operatorname{rank} C_1 = \operatorname{rank} C_1'$.

Our aim is to study the equivalence classes in (2.5) and the variation of their equivalence invariants (for example, the block similarity invariants of the pair (C, A) or those of its restriction (C_1, A_1)) when the pair (C, A) is perturbed in such a way that $S = \mathbb{C}^d \times \{0\} \subset \mathbb{C}^N$ is preserved as a conditioned invariant subspace, and mainly when it is marked.

We will use Arnold's techniques of the so-called versal deformations (that is, canonical forms of local differentiable families of perturbations). The starting point is the fact that the corresponding equivalence classes are orbits under the action of a Lie group, and hence they are submanifolds. Then versal/miniversal deformations can be obtained as submanifolds which are transverse/minitransverse to the orbit.

Definition 2.7 *Let $\mathcal{M}$ be a manifold. A* deformation *of $a \in \mathcal{M}$ is a differentiable map*

$$\varphi : \Lambda \longrightarrow \mathcal{M},$$

where Λ is a neighborhood of the origin in $\mathbb{C}^l$ and $\varphi(0) = a$. The image $\varphi(\Lambda)$ is said to be a family of deformations *of $a \in \mathcal{M}$.*

If there is a Lie group $\mathcal{G}$ acting on the differentiable manifold $\mathcal{M}$,

$$\begin{aligned} \mathcal{G} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (p, a) &\longmapsto p * a, \end{aligned}$$

a deformation is called "versal" if any other deformation is induced from it in the following sense:

Definition 2.8 *Let $\mathcal{M}$ be a manifold and $\mathcal{G}$ a Lie group acting on it. A deformation of $a \in \mathcal{M}$, $\varphi : \Lambda \longrightarrow \mathcal{M}$ is called* versal *if, given any other deformation of $a \in \mathcal{M}$, $\psi : \Gamma \longrightarrow \mathcal{M}$, there is a neighborhood of the origin $\Gamma' \subset \Gamma$, a differentiable map $\rho : \Gamma' \longrightarrow \Lambda$ and a deformation of the identity $I \in \mathcal{G}$, $\delta : \Gamma' \longrightarrow \mathcal{G}$ such that*

$$\psi(\tau) = \delta(\tau) * \varphi(\rho(\tau)) \, , \forall \tau \in \Gamma'.$$

It is called miniversal *if it has the minimal dimension among the versal deformations.*

Remark 2.9 *It is enough to compute a miniversal deformation of a point of the orbit; then, a miniversal deformation of any other point of the same orbit is induced from it by means of the group action.*

As in the Arnold case, the "closed orbit lemma " ([24], p. 37) will ensure that the equivalence classes (in fact, the orbits) are differentiable manifolds.

Proposition 2.10 *If $\mathcal{G}$ is an algebraic group, for all $a \in \mathcal{M}$, the orbit $\mathcal{O}_a = \{p{*}a : p \in \mathcal{G}\}$ under the action of $\mathcal{G}$ is a submanifold of $\mathcal{M}$ locally closed, where the boundary is the union of orbits of strictly smaller dimension.*

We now recall the key relation between "versality " and "transversality ".

Definition 2.11 *Let $\mathcal{N} \subset \mathcal{M}$ be a submanifold of the manifold $\mathcal{M}$ and $\varphi : \Lambda \longrightarrow \mathcal{M}$ be a differentiable map. For $0 \in \Lambda$, φ is said to be* transverse *to $\mathcal{N}$ in 0 if $\varphi(0) \in \mathcal{N}$ and the tangent space to $\mathcal{M}$ at the point $\varphi(0)$ verifies*

$$T_{\varphi(0)}\mathcal{M} = Im\, d\varphi_0 + T_{\varphi(0)}\mathcal{N}.$$

φ(or L) is said to be minitransverse *if the sum is a direct sum.*

As pointed out above, the key point is the following proposition, proved in [1] for square matrices, and which can be generalized (for example [24]) to the cases like the above one, where the equivalence classes are submanifolds given as orbits under the action of a Lie group.

Proposition 2.12 *A deformation $\varphi : \Lambda \longrightarrow \mathcal{M}$ of $a \in \mathcal{M}$ is versal/miniversal if and only if it is transverse/minitransverse to the orbit $\mathcal{O}_a$ at the origin $0 \in \Lambda$.*

Corollary 2.13 *In the conditions of 2.8, if $\gamma : \Gamma \longrightarrow \mathcal{M}$, $\Gamma \subset \mathbb{C}^\sigma$, is a local parameterization of $\mathcal{M}$ with $\gamma(\tilde{a}) = a$, and $\{e_1, \dots, e_l\}$ is a basis of a supplementary subspace of $T_{\tilde{a}}(\gamma^{-1}(\mathcal{O}_a))$ in $\mathbb{C}^\sigma$, then a miniversal deformation of $a \in \mathcal{M}$ is $\varphi : \Lambda \longrightarrow \mathcal{M}$ defined by*

$$\varphi(\lambda_1, \dots, \lambda_l) = \gamma(\tilde{a} + \lambda_1 e_1 + \cdots + \lambda_l e_l),$$

where Λ is small enough to assure $\varphi(\Lambda) \subset \Gamma$.

Moreover, the codimension of $\mathcal{O}_a$ is l.

3 Pairs of matrices having a (C, A)-invariant fixed subspace

In this section, we characterize the elements of the set $\mathcal{M}$ formed by the pairs of matrices having the subspace $\mathbb{C}^d \times \{0\} \subset \mathbb{C}^N$ as a conditioned invariant. Moreover, we see that it is a stratified manifold and finally we describe a Lie group that acts on $\mathcal{M}$ in such a way that the orbits are just the equivalence classes in (2.5).

Definition 3.1 *Let*
$\mathcal{M} = \{(C, A) \in M_{N+n}(\mathbb{C}) \times M_N(\mathbb{C}) : A(S \cap \operatorname{Ker} C) \subset S, \quad S = \mathbb{C}^d \times \{0\} \subset \mathbb{C}^N\}$,

$$\mathcal{M}_r = \left\{ \begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \\ \hline C_1 & C_2 \end{pmatrix} \in \mathcal{M} : \operatorname{rank} C_1 = r \right\}, \qquad \mathcal{M} = \bigcup_{0 \le r \le r^*} \mathcal{M}_r, \text{ where } r^* = min(d, n).$$

Now we characterize the pairs in $\mathcal{M}$.

Proposition 3.2 *A pair $(C, A) \in M_{N+n}(\mathbb{C}) \times M_N(\mathbb{C})$ belongs to $\mathcal{M}$ if and only if its d-block-partitioned form can be written, for some $F \in M_{N-d,n}(\mathbb{C})$, as*

$$\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} A_1 & A_3 \\ FC_1 & A_2 \\ \hline C_1 & C_2 \end{pmatrix}.$$

Proof.

If $a \in \mathcal{M}$, from definition 2.2 it holds that

$$A((\mathbb{C}^d \times \{0\}) \cap \operatorname{Ker} C) \subset \mathbb{C}^d \times \{0\}. \quad (*)$$

The elements of $(\mathbb{C}^d \times \{0\}) \cap \operatorname{Ker} C$ have the form $\begin{pmatrix} x \\ 0 \end{pmatrix}$ with $C_1 x = 0$. Thus, $(*)$ is verified if and only if $C_1 x = 0$ implies $A_4 x = 0$, that is, $\operatorname{Ker} C_1 \subset \operatorname{Ker} A_4$. But this last condition is equivalent to $\operatorname{Im} A_4^t \subset \operatorname{Im} C_1^t$; or, equivalently, $A_4^t = C_1^t F^t$ for some $F \in M_{N-d,n}(\mathbb{C})$. ∎

In order to study the differentiable structure of $\mathcal{M}$, let us consider the set

$$\mathcal{N} = \{(DG, D) : D \in M_{d,n}(\mathbb{C}), G \in M_{n,l}(\mathbb{C})\}.$$

In general, it is not a manifold. For example, $\{(xy, x) : x \in \mathbb{R}, y \in \mathbb{R}\} = \{(0, 0)\} \cup \{(z, x) : x \neq 0, z \in \mathbb{R}\}$. Let us see that we can stratify $\mathcal{N}$ by means of $\operatorname{rank} D$:

Proposition 3.3 *The set*

$$\mathcal{N}_r = \{(DG, D) : D \in M_{d,n}(\mathbb{C}), G \in M_{n,l}(\mathbb{C}), \operatorname{rank} D = r\}$$

is a manifold of dimension $r(l+n+d-r)$.

Proof.

If $D_0 \in M_{d,n}(\mathbb{C})$ with $\operatorname{rank} D_0 = r$, there exists an open set containing D_0, $\mathcal{U}_{D_0} \subset M^r_{d,n}(\mathbb{C})$, where $M^r_{d,n}(\mathbb{C})$ denotes the set of matrices of $M_{d,n}(\mathbb{C})$ with $\operatorname{rank} r$ which from [3] is a manifold of dimension $dn - (d-r)(n-r)$. Take $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq d$ such that the rows $d_{i_j} = (d_{i_j,1}, d_{i_j,2}, \ldots, d_{i_j,n})$ $j = 1, 2, \ldots, r$ of D determine a basis $\mathcal{D}$ of $\operatorname{Im} D^t$ for $D \in \mathcal{U}_{D_0}$ (shrinking it if necessary).

Let $\Pi : \mathbb{C}^n \longrightarrow \operatorname{Im} D^t$ be the orthogonal projection onto $\operatorname{Im} D^t = (\ker D)^\perp$. If we denote by g^i the i column of the matrix G, $K_D = (d^t_{i_1}, \ldots, d^t_{i_r}) \in M_{n,r}(\mathbb{C})$ and $J \in M_{r,l}(\mathbb{C})$ is the matrix where the i column is formed by the components of $\Pi(g^i)$ in the basis $\mathcal{D}$ of $\operatorname{Im} D^t$, it is obvious that

$$DG = D(\Pi(g^1), \ldots, \Pi_0(g^l)) = DK_D J \text{ for } D \in \mathcal{U}_{D_0}. \qquad (*)$$

We consider the map

$$\varphi : M_{r,l}(\mathbb{C}) \times \mathcal{U}_{D_0} \longrightarrow \mathcal{N}_r, \quad \varphi(J, D) = (DK_D J, D) = (DG, D).$$

This map φ is C^∞ because the elements of $DK_D J$ are polynomials of the elements of J and D, it is injective because the rank of $DK_D = (< d_i, d_{i_j} >)_{i,j}$ is r, and (*) implies that $\operatorname{Im} \varphi = \{(DG, D) : D \in \mathcal{U}_{D_0}, G \in M_{n,l}(\mathbb{C})\}$.

Moreover, $d\varphi_{(J,D)}$ is injective because $d\varphi_{(J,D)}(\dot{J}, \dot{D}) = (\dot{D}K_D J + D\dot{K}_D J + DK_D \dot{J}, \dot{D}) = 0$ implies $\dot{D} = 0$, $\dot{K}_D = 0$ ($\dot{K}_D$ is a submatrix of $\dot{D}$) and $\dot{J} = 0$. In brief, φ and its inverse φ^{-1} defined in $\operatorname{Im} \varphi$ are C^∞.

Finally, we prove that if $\mathcal{U}_{D_1}$ is another open neighborhood of the same kind verifying $\mathcal{U}_{D_1} \cap \mathcal{U}_{D_0} \neq \varnothing$ and the restriction of the corresponding maps φ and ψ on $M_{r,l}(\mathbb{C}) \times (\mathcal{U}_{D_1} \cap \mathcal{U}_{D_0})$, respectively, is considered, then the composition $\psi^{-1} \circ \varphi$ is differentiable. If we denote the matrices K_D corresponding to φ and ψ by K^0_D and K^1_D, respectively, we have $K^0_D = K^1_D S$ with S an invertible matrix because it is the matrix of a change of basis in $\operatorname{Im} D^t$. Hence,

$$(\psi^{-1} \circ \varphi)(J, D) = \psi^{-1}(DK^0_D J, D) = \psi^{-1}(DK^1_D SJ, D) = (SJ, D).$$

Then, the pairs $(\varphi, M_{r,l}(\mathbb{C}) \times \mathcal{U}_{D_0})$ are a coordinate system and provide $\mathcal{N}_r$ with a manifold structure of dimension $r(l+n+d-r)$. ■

Therefore, $\mathcal{M} = \bigcup_r \mathcal{M}_r$ is a stratified differentiable manifold:

Theorem 3.4 *With the notation in 3.1, $\mathcal{M}_r$ is a manifold of dimension*

$$\sigma_r = d^2 + (N-d)(N+n) + r(N+n-r).$$

In fact, a local parameterization is

$$\gamma : M_d(\mathbb{C}) \times M_{N-d}(\mathbb{C}) \times M_{d,N-d}(\mathbb{C}) \times \mathcal{U}_{C_1} \times M_{n,N-d}(\mathbb{C}) \times M_{r,N-d}(\mathbb{C}) \longrightarrow \mathcal{M}_r$$

$$\gamma(A_1, A_2, A_3, C_1, C_2, J) = \left(\begin{array}{cc} A_1 & A_3 \\ JK^t_{C_1^t} C_1 & A_2 \\ \hline C_1 & C_2 \end{array}\right).$$

Notation 3.5 *From now on we will indicate the coordinates of any $a \in \mathcal{M}_r$ by $\tilde{a}$; namely*

$$\tilde{a} = \gamma^{-1}(a) = (A_1, A_2, A_3, C_1, C_2, J) \in \mathbb{C}^{\sigma_r}.$$

Following (2.5), we restrict to $\mathcal{M}$ in a natural way the usual change of basis in the manifold $M_{N+n,N}(\mathbb{C})$. Now the chain of subspaces $\mathbb{C}^d \times \{0\} \subset \mathbb{C}^N \times \{0\} \subset \mathbb{C}^{N+n}$ must be preserved, so we consider:

Definition 3.6 *Let $\mathcal{G} \subset Gl(N+n)$ be defined by*

$$\mathcal{G} = \left\{ p = \left(\begin{array}{cc|c} P_1 & P_3 & R_1 \\ 0 & P_2 & R_2 \\ \hline 0 & 0 & Q \end{array}\right) \in Gl(N+n), P_1 \in Gl(d), P_2 \in Gl(N-d), Q \in Gl(n) \right\}.$$

It is a straightforward computation that $\mathcal{G}$ is a subgroup of $Gl(N+n)$ and that:

Proposition 3.7 *The natural action of the subgroup $\mathcal{G} \subset Gl(N+n)$ on the differentiable manifold $M_{N+n,N}(\mathbb{C})$ can be restricted to $\mathcal{M}_r$:*

$$\mathcal{G} \times \mathcal{M}_r \longrightarrow \mathcal{M}_r, \quad (p, a) \longmapsto p * a;$$

that is,

$$p * a = \left(\begin{array}{cc|c} P_1 & P_3 & R_1 \\ 0 & P_2 & R_2 \\ \hline 0 & 0 & Q \end{array}\right) \left(\begin{array}{cc} A_1 & A_3 \\ FC_1 & A_2 \\ \hline C_1 & C_2 \end{array}\right) \begin{pmatrix} P_1 & P_3 \\ 0 & P_2 \end{pmatrix}^{-1} \in \mathcal{M}_r.$$

Proposition (2.6) shows that:

Proposition 3.8 *If we denote by $\mathcal{O}_a$ the orbit of $a \in \mathcal{M}_r$ under the action of $\mathcal{G}$, then $\mathcal{O}_a$ is the class of a with regard to the equivalence relation in (2.5).*

4 Miniversal deformation preserving a (C,A)-invariant subspace

We restrict ourselves to the stratum of $\mathcal{M}$ where the rank of C_1 is maximal:

$$\mathcal{M}^* = \{a \in \mathcal{M} : \operatorname{rank} C_1 = r^*\},$$

where $r^* = \min(d, n)$. Note that, in fact, this is the generic situation:

Proposition 4.1 *$\mathcal{M}^*$ is an open dense subset in $\mathcal{M}$.*

Proof.

It is obvious that $\mathcal{M}^*$ is an open set. To prove the second property, it is sufficient to point out that for any C_1 and $\varepsilon > 0$ there is C_1', with $\operatorname{rank} C_1 = r^*$, such that $\|C_1' - C_1\| < \varepsilon$, and hence

$$\| \left(\begin{array}{cc} A_1 & A_3 \\ FC_1' & A_2 \\ \hline C_1' & C_2 \end{array}\right) - \left(\begin{array}{cc} A_1 & A_3 \\ FC_1 & A_2 \\ \hline C_1 & C_2 \end{array}\right) \| \leq \|FC_1' - FC_1\| + \|C_1' - C_1\| < \varepsilon(1 + \|F\|). \qquad \blacksquare$$

Thus, we have our first main result:

Theorem 4.2 *Let $a \in \mathcal{M}^*$ with $\tilde{a} = (A_1, A_2, A_3, C_1, C_2, 0) \in \mathbb{C}^\sigma$.*
A miniversal deformation of this point in $\mathcal{M}$ is given by the set of matrices

$$\left(\begin{array}{cc} A_1 + X_1 & A_3 + X_3 \\ Z(C_1 + Y_1) & A_2 + X_2 \\ \hline C_1 + Y_1 & C_2 + Y_2 \end{array}\right)$$

satisfying the conditions

(1) $X_2 C_2^* + Z = 0$,
(2) $Y_1 C_1^* + Y_2 C_2^* = 0$,
(3) $X_1 C_1^* + X_3 C_2^* = 0$,
(4) $X_3 A_2^* - A_1^* X_3 - C_1^* Y_2 = 0$,
(5) $X_1 A_1^* + X_3 A_3^* - A_1^* X_1 - C_1^* Y_1 = 0$,
(6) $X_2 A_2^* - A_3^* X_3 - A_2^* X_2 - C_2^* Y_2 = 0$.

Proof.

Let $V \subset \mathcal{M}^*$ a small enough neighborhood of a in the linear variety defined by $(1) - \ldots -$ (6). It is sufficient to prove that V is minitransversal to $\mathcal{O}_a$ at a or, equivalently, that $\gamma^{-1}(V)$ is minitransversal to $\gamma^{-1}(\mathcal{O}_a)$ at $\tilde{a}$.

Notice that in the proof of (3.3), if D has maximal rank, then $DG \neq 0$ if $G \neq 0$. Therefore, for the maximal stratum $\mathcal{M}^*$, the parameterization in (3.4) can be simplified taking

$\mathcal{U}_{C_1} \subset M_{n,d}(\mathbb{C})$ and

$$\gamma : \mathbb{C}^\sigma \longrightarrow \mathcal{M}^*, \quad \sigma = d^2 + (N-d)(N+n) + r^*(N+n-r^*)$$

$$\gamma(A_1, A_2, A_3, C_1, C_2, F) = \begin{pmatrix} A_1 & A_3 \\ FC_1 & A_2 \\ \hline C_1 & C_2 \end{pmatrix}.$$

If we consider the $\mathcal{G}$-action in $\mathbb{C}^\sigma$ induced in a natural way by (3.7), it is a straightforward computation that

$$p*\tilde{a} = (P_1A_1P_1^{-1} + P_3FC_1P_1^{-1} + R_1C_1P_1^{-1}, -(P_2FC_1 + R_2C_1)P_1^{-1}P_3P_2^{-1} + (P_2A_2 + R_2C_2)P_2^{-1},$$

$$-(P_1A_1 + P_3FC_1 + R_1C_1)P_1^{-1}P_3P_2^{-1} + (P_1A_3 + P_3A_2 + R_1C_2)P_2^{-1}, QC_1P_1^{-1},$$
$$-QC_1P_1^{-1}P_3P_2^{-1} + QC_2P_2^{-1}, P_2FQ^{-1} + R_2Q^{-1}).$$

Hence $\gamma^{-1}(\mathcal{O}_a) = \mathcal{O}_{\tilde{a}} = \{p * \tilde{a}, p \in \mathcal{G}\}$. To conclude the theorem let us see that $\gamma^{-1}(V)$ is, locally, just the normal variety $\tilde{a} + (T_{\tilde{a}}\mathcal{O}_{\tilde{a}})^\perp$, where $T_{\tilde{a}}\mathcal{O}_{\tilde{a}}$ is the tangent space to $\mathcal{O}_{\tilde{a}}$ at $\tilde{a}$. Clearly, this tangent space is $\operatorname{Im} d\alpha_I$, where $d\alpha_I$ is the derivative at the identity $I \in \mathcal{G}$ of the map $\alpha : \mathcal{G} \longrightarrow \mathbb{C}^\sigma, \quad \alpha(p) = p * \tilde{a}$. Deriving this map we have:

$$d\alpha_I(\dot{p}) = (\dot{P}_1A_1 - A_1\dot{P}_1 + \dot{R}_1C_1, \dot{P}_2A_2 + \dot{R}_2C_2 - A_2\dot{P}_2, -A_1\dot{P}_3 + \dot{P}_1A_3 + \dot{P}_3A_2 + \dot{R}_1C_2 - A_3\dot{P}_2,$$

$$\dot{Q}C_1 - C_1\dot{P}_1, -C_1\dot{P}_3 + \dot{Q}C_2 - C_2\dot{P}_2, \dot{R}_2)$$

for any $\dot{p}$ belonging to $T_I\mathcal{G}$, that is, for $\dot{P}_1 \in M_d(\mathbb{C}), \dot{P}_2 \in M_{N-d}(\mathbb{C}), \dot{P}_3 \in M_{d,N-d}(\mathbb{C}), \dot{R}_1 \in M_{d,n}(\mathbb{C}), \dot{R}_2 \in M_{N-d,n}(\mathbb{C}), \dot{Q} \in M_n(\mathbb{C})$.

Then, $\tilde{x} = (X_1, X_2, X_3, Y_1, Y_2, Z) \in (T_{\tilde{a}}\mathcal{O}_{\tilde{a}})^\perp$ if and only if, for any $\dot{P}_1, \dot{P}_2, \dot{P}_3, \dot{R}_1, \dot{R}_2, \dot{Q}$ as above,

$$\operatorname{trace}(X_1^*(\dot{P}_1A_1 - A_1\dot{P}_1 + \dot{R}_1C_1)) + \operatorname{trace}(X_2^*(\dot{P}_2A_2 + \dot{R}_2C_2 - A_2\dot{P}_2)) +$$

$$+ \operatorname{trace}(X_3^*(-A_1\dot{P}_3 + \dot{P}_1A_3 + \dot{P}_3A_2 + \dot{R}_1C_2 - A_3\dot{P}_2)) + \operatorname{trace}(Y_1^*(\dot{Q}C_1 - C_1\dot{P}_1)) +$$
$$+ \operatorname{trace}(Y_2^*(-C_1\dot{P}_3 + \dot{Q}C_2 - C_2\dot{P}_2)) + \operatorname{trace}(Z^*\dot{R}_2) = 0$$

or, equivalently,

$$\operatorname{trace}(X_1A_1^* + X_3A_3^* - A_1^*X_1 - C_1^*Y_1)\dot{P}_1^* + \operatorname{trace}(X_3A_2^* - A_1^*X_3 - C_1^*Y_2)\dot{P}_3^* +$$

$$+ \operatorname{trace}(X_1C_1^* + X_3C_2^*)\dot{R}_1^* + \operatorname{trace}(X_2A_2^* - A_3^*X_3 - A_2^*X_2 - C_2^*Y_2)\dot{P}_2^* +$$
$$+ \operatorname{trace}(X_2C_2^* + Z)R_2^* + \operatorname{trace}(Y_1C_1^* + Y_2C_2^*)\dot{Q}^* = 0.$$

Hence, $\tilde{x} \in (T_{\tilde{a}}\mathcal{O}_{\tilde{a}})^\perp$ if and only if (1) $- \ldots -$ (6) are verified. ■

In the particular case when $A_3 = 0$ and $C_2 = 0$, then the equations in Theorem 4.2 become just those obtained in [12] with regard to the general block similarity. Hence, we have:

Corollary 4.3 *Let us consider the particular case of a BK-matrix $a_{BK} \in \mathcal{M}^*$ and $S = \mathbb{C}^d \times \{0\}$ a supplementary of the unobservable subspace. Then, a miniversal deformation of a_{BK} is*

$$\left(\begin{array}{cc} N & 0 \\ 0 & J \\ \hline E & 0 \end{array}\right) + \left(\begin{array}{cc} 0 & X_3^{FRW} \\ 0 & X_2^{END} \\ \hline Y_1^{OBS} & 0 \end{array}\right),$$

where

- *(i)* *The deformation parameters in the i-row of Y_1^{OBS} $(2 \leq i \leq r)$ are placed in the first $(i-1)$ blocks as follows: $k_i + 1, \ldots, k_1 - 1; \ldots; k_1 + \ldots + k_{j-1} + k_i + 1, \ldots, k_1 + \ldots + k_j - 1; \ldots; k_1 + \ldots + k_{i-2} + k_i + 1, \ldots, k_1 + \ldots + k_{i-1} - 1$. (Notice that there are no parameters in the j-block if $k_j \leq k_i + 1$.)*
- *(ii)* *The deformation parameters in X_2^{END} are those of the miniversal deformation of the square matrix J in [1].*
- *(iii)* *The deformation parameters in X_3^{FRW} are just the entries of the rows: $1, k_1 + 1, k_1 + k_2 + 1, \ldots, k_1 + k_2 + \ldots + k_{r-1} + 1$.*

Example 4.4 For a BK-matrix with BK-indices $(3, 3, 2, 1)$ and J a nilpotent matrix with Segre characteristic $(1, 4, 2)$, we have

$$\left[\begin{array}{ccc|ccc|cc|c||c|cccc|cc}
 & & & & & & & & & * & * & * & * & * & * & * \\
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\ \hline
 & & & & & & & & & * & * & * & * & * & * & * \\
 & & & 1 & & & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & & & \\ \hline
 & & & & & & & & & * & * & * & * & * & * & * \\
 & & & & & & 1 & & & & & & & & & \\ \hline
 & & & & & & & & & * & * & * & * & * & * & * \\ \hline\hline
 & & & & & & & & & * & & & & * & & * \\ \hline
 & & & & & & & & & * & * & * & * & * & * & * \\
 & & & & & & & & & & 1 & & & & & \\
 & & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & & & 1 & & & \\ \hline
 & & & & & & & & & * & & & * & * & * & * \\
 & & & & & & & & & & & & & & 1 & \\ \hline\hline
 & & 1 & & & & & & & & & & & & & \\ \hline
 & & & & & 1 & & & & & & & & & & \\ \hline
 & & & & & & & 1 & & & & & & & & \\ \hline
 & * & & & * & & & & 1 & & & & & & &
\end{array}\right]$$

Remark 4.5 *The above corollary shows that Theorem (4.2) generalizes the results in [12] when the preserved conditioned invariant subspace is not necessarily a supplementary of the unobservable one. In this sense, the notations Y_1^{OBS} and X_2^{END} point out that the particular cases of an observable pair and a square matrix are included, with $d = N$ and $d = 0$, respectively.*

5 Miniversal deformation of observable marked matrices

As pointed out in the introduction, we will solve the equations in theorem (4.2) explicitly in those cases where $a \in \mathcal{M}^*$ is an observable marked (see definition (2.4)) matrix (then $r^* = n$). It can be easily observed (see [5] for further explanation) that if $a \in \mathcal{M}^*$ is observable, $a \in \mathcal{M}^*$ is marked if and only if there exists a matrix $a_c \in \mathcal{O}_a$ of the form described in the following definition which we will call its canonical form.

Definition 5.1 *Let*

$$a = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \\ \hline C_1 & 0 \end{pmatrix} \in \mathcal{M}^*$$

be an observable marked matrix with $q = (q_1, \dots, q_n)$ and $p = (p_1, \dots, p_n)$ being the BK-indices of the pairs (C_1, A_1) and (C, A), respectively, verifying $q_1 \geq q_2 \geq \cdots \geq q_n$, $p_i \geq q_i$, $q_1 + \cdots + q_n = d$, $p_1 + \cdots + p_n = N$, and let us define $\delta_i = p_i - q_i$. Notice that $(p_1, \dots, p_n)$ are not necessarily in non increasing order.

(i) We say that a is in canonical form *(and then we write a_c) if*

- *(1) (C_1, A_1) is a Brunovsky pair.*
- *(2) $A_2 = diag(N_{\delta_1}, \dots, N_{\delta_n}) \in M_{N-d}(\mathbb{C})$, where only $\delta_i \neq 0$ are considered.*
- *(3) $A_3 = (A_{3,i,j})_{1 \leq i,j \leq n}$, $A_3 \in M_{d,N-d}(\mathbb{C})$, $A_{3,i,j} \in M_{q_i,\delta_j}(\mathbb{C})$*

$$A_{3,i,i} = \begin{bmatrix} e_{\delta_i} \\ 0 \end{bmatrix}$$ *if $1 \leq i \leq m$, $A_{3,i,j} = 0$ otherwise, where only $\delta_i \neq 0$ are considered, and then e_{δ_i} will be a row matrix of zeros and one 1 in the position δ_i with the size corresponding to the context.*

(ii) We say that all the matrices equivalent to the preceding one are of type (q, p).

Example 5.2 The following matrix is an observable marked matrix in canonical form of type $((3, 3, 2, 1), (4, 3, 6, 3))$:

$$
\left[\begin{array}{ccc|ccc|cc|c||c|cccc|cc}
&&&&&&&&&1&&&&&& \\
1&&&&&&&&&&&&&&& \\
&1&&&&&&&&&&&&&& \\
\hline
&&&&&&&&&&&&&&& \\
&&&1&&&&&&&&&&&& \\
&&&&1&&&&&&&&&&& \\
\hline
&&&&&&&&&&&&&1&& \\
&&&&&&1&&&&&&&&& \\
\hline
&&&&&&&&&&&&&&&1 \\
\hline\hline
&&&&&&&&&&&&&&& \\
\hline
&&&&&&&&&&&&&&& \\
&&&&&&&&&&1&&&&& \\
&&&&&&&&&&&1&&&& \\
&&&&&&&&&&&&1&&& \\
\hline
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&1& \\
\hline\hline
&&1&&&&&&&&&&&&& \\
\hline
&&&&&1&&&&&&&&&& \\
\hline
&&&&&&&1&&&&&&&& \\
\hline
&&&&&&&&1&&&&&&&
\end{array}\right]
$$

Remark 5.3 *Notice that the matrices in (5.1) appear in the miniversal deformation in (4.3) when J is nilpotent. For instance, the one in (5.2) appears in the miniversal deformation in (4.4). In general, they are perturbations making a_{BK} observable, with $Y_1^{OBS} = 0$ and $X_2^{END} = 0$. In fact, it can be seen that, somehow, they are "minimal" observable deformations of a_{BK}, that is to say, pairs in (4.3) having a minimal number of non-zero deformation parameters to be observable.*

When we solve the set of equations in theorem (4.2), the following special types of Toeplitz matrices often appear:

Definition 5.4 *(1) We say that a matrix $X = (x_{i,j}) \in M_{\gamma,\beta}(\mathbb{C})$ is a* T-matrix *if it is a Toeplitz matrix; that is, if it is constant along the diagonals.*

(2) If X is a T-matrix such that $x_{i,1} = 0$ if $i > 1$, we say that X is an UT-matrix *(upper Toeplitz matrix).*

(3) We say that a block matrix $X = \begin{bmatrix} X_{i,j} \end{bmatrix}_{1\le i\le r, 1\le j\le s}$, $X_{i,j} \in M_{\gamma_i,\beta_j}(\mathbb{C})$ is a block T-matrix *if each block $X_{i,j}$ is a T-matrix. We define a* block UT-matrix *analogously.*

We now solve equations $(1) - \ldots - (6)$ in Theorem (4.2) when the matrix $a_c \in \mathcal{M}^*$ is an observable marked matrix in canonical form. The following theorem describes the corresponding solutions:

Theorem 5.5 (First Miniversal Deformation) *Let $a_c \in \mathcal{M}^*$ be an observable marked matrix in canonical form of type (q,p) as in (5.1).*

Then, a miniversal deformation of $a_c \in \mathcal{M}^*$ is given by the set of matrices

$$
a_c + \left(\begin{array}{cc} X_1 & X_3 \\ 0 & X_2 \\ \hline Y_1 & Y_2 \end{array}\right)
$$

such that

(2') $(Y_{1,h,k})_{q_k} = 0$,
(3') $(X_{1,h,k})_{q_k} = 0$,
(4') $\begin{bmatrix} X_{3,h,k} \\ Y_{2,h,k} \end{bmatrix}$ *UT-matrix,*
(5') $\begin{bmatrix} (X_{3,h,k})_{\delta_k} & X_{1,h,k} \\ 0 & Y_{1,h,k} \end{bmatrix}$ *T-matrix if* $\delta_k > 0$,
$\begin{bmatrix} X_{1,h,k} \\ Y_{1,h,k} \end{bmatrix}$ *UT-matrix if* $\delta_k = 0$,
(6') $\begin{bmatrix} X_{2,h,k} \\ (X_{3,h,k})^1 \end{bmatrix}$ *UT-matrix provided that* $\delta_h > 0$ *and* $\delta_k > 0$,

where $(\cdot)_\nu$, $(\cdot)^\nu$ *mean, respectively, the* ν *column/row of* $(\cdot)$.

Proof.

We will denote by $(\cdot)_{\widehat{\nu}}$ and $(\cdot)^{\widehat{\nu}}$ the matrix $(\cdot)$ from which the ν column/row, respectively, has been removed.

Solving each equation in (4.2), we have:

(1) $X_2 C_2^* + Z = 0$
Using that $C_2 = 0$ (from definition (5.1)), it turns out that $Z = 0$.
(2) $Y_1 C_1^* + Y_2 C_2^* = 0$
Also using that $C_2 = 0$ and the decomposition into blocks of the matrices, we have

$$0 = \sum_{i=1}^{m} Y_{1,h,i} C_{1,k,i}^* \ , \quad 1 \le h, k \le n$$

and using the form of the blocks of C_1, we obtain $0 = Y_{1,h,k} C_{1,k,k}^*$,. Finally, using the introduced notation, this equation is equivalent $(Y_{1,h,k})_{q_k} = 0$.
(3) $X_1 C_1^* + X_3 C_2^* = 0$
This equation is the same as (2), but the matrices involved are now $X_1 \in M_d(\mathbb{C})$, $X_3 \in M_{d,N-d}(\mathbb{C})$.
(4) $X_3 A_2^* - A_1^* X_3 - C_1^* Y_2 = 0$
Using the decomposition into blocks of the matrices, we have

$$\sum_{i=1}^{n} X_{3,h,i} A_{2,k,i}^* - \sum_{j=1}^{m} A_{1,j,h}^* X_{3,j,k} - \sum_{\ell=1}^{n} C_{1,\ell,h}^* Y_{2,\ell,k} = 0 \, , 1 \le h, k \le n.$$

Considering the form of the matrices A_1, A_2 and C_1, we have $X_{3,h,k} N_{\delta_k}^* - N_{q_k}^* X_{3,h,k} - e_{q_h}^* Y_{2,h,k} = 0$; and using the above notation, we obtain $\begin{bmatrix} 0 \ (X_{3,h,k})_{\widehat{\delta_k}} \end{bmatrix} = \begin{bmatrix} (X_{3,h,k})^{\widehat{1}} \\ Y_{2,h,k} \end{bmatrix}$, which is equivalent to (4').

(5) $X_1A_1^* + X_3A_3^* - A_1^*X_1 - C_1^*Y_1 = 0$
Using the decomposition into blocks, the last equation is equal to

$$\sum_{i=1}^{m} X_{1,h,i}A_{1,k,i}^* + \sum_{i=1}^{n} X_{3,h,i}A_{3,k,i}^* = \sum_{i=1}^{m} A_{1,i,h}^*X_{1,i,k} + \sum_{i=1}^{n} C_{1,i,h}^*Y_{1,i,k}\,,\, 1 \leq h,k \leq n.$$

Considering the form of the matrices, we distinguish two cases:

(a) If $\delta_k > 0$, $\quad X_{1,h,k}N_{q_k}^* + X_{3,h,k}\begin{bmatrix} e_{\delta_k} \\ 0 \end{bmatrix}^* = N_{q_h}^*X_{1,h,k} + e_{q_h}^*Y_{1,h,k}\,.$

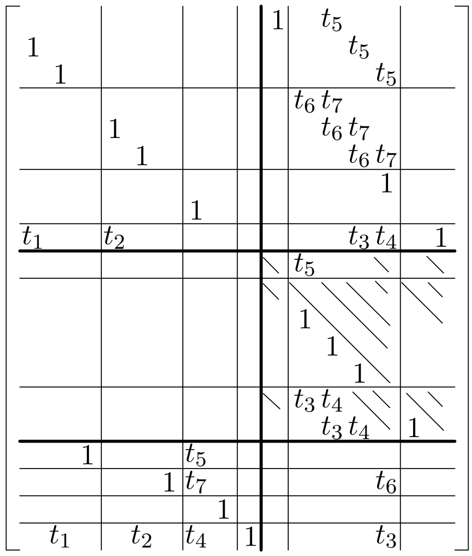
Using the above notation, we have $\left[(X_{3,h,k})_{\delta_k}, (X_{1,h,k})_{\widehat{q_k}}\right] = \begin{bmatrix} (X_{1,h,k})^{\widehat{1}} \\ Y_{1,h,k} \end{bmatrix}$, which is equivalent to (5').

(b) If $\delta_k = 0$, $\quad X_{1,h,k}N_{q_k}^* = N_{q_h}^*X_{1,h,k} + e_{q_h}^*Y_{1,h,k}\,.$

Using the above notation, we have $\left[0, (X_{1,h,k})_{\widehat{q_k}}\right] = \begin{bmatrix} (X_{1,h,k})^{\widehat{1}} \\ Y_{1,h,k} \end{bmatrix}$. Therefore, the solution for this case is equivalent to considering $(X_{3,h,k})_{\delta_k} = 0$ in the general case, and this gives us (5').

(6) $X_2A_2^* - A_3^*X_3 - A_2^*X_2 - C_2^*Y_2 = 0$
Using the decomposition into blocks of the matrices and that $C_2 = 0$, we have
$\sum_{i=1}^{n} X_{2,h,i}A_{2,k,i}^* - \sum_{i=1}^{m} A_{3,i,h}^*X_{3,i,k} - \sum_{i=1}^{n} A_{2,i,h}^*X_{2,i,k} = 0\,,\, 1 \leq h,k \leq n, \quad \delta_h > 0, \delta_k > 0.$
Considering the form of the matrices, we have $X_{2,h,k}N_{\delta_k}^* - A_{3,h,h}^*X_{3,h,k} - N_{\delta_h}^*X_{2,h,k} = 0$. This equation is equivalent to $\left[0, (X_{2,h,k})_{\widehat{\delta_k}}\right] = \begin{bmatrix} (X_{2,h,k})^{\widehat{1}} \\ (X_{3,h,k})^{1} \end{bmatrix}$, and we obtain (6'). ■

Example 5.6 Given the observable marked matrix in canonical form of type $((3,3,2,1),(4,3,6,3))$ in example (5.2), its miniversal deformation given by theorem (5.5) is



where all the parameters appear along the indicated diagonals and t_i are the parameters appearing in more than one block in X_1, X_2, X_3, Y_1, Y_2.

As an application we will compute codim $\mathcal{O}_{a_c}$ in (6.1). In fact, we use it to derive a new miniversal deformation of $a_c \in \mathcal{M}^*$ without repeated parameters, which will be more useful to study the effect of each parameter. We construct it by taking an appropriate basis of a suitable supplementary subspace of $T_{\tilde{a}_c}\mathcal{O}_{\tilde{a}_c}$.

Definition 5.7 *Let $a_c \in \mathcal{M}^*$ be an observable marked matrix in canonical form (see definition (5.1)). We define the elements a^i_{2hk}, a^i_{3hk} and c^i_{1hk} in $\mathbb{C}^\sigma$ having the same block sizes as in $\tilde{a}_c \in \mathcal{M}^*$, and all the entries 0 except one 1 placed in the first row of the block A_{2hk}, A_{3hk} or C_{1hk}, respectively, and in their i-column.*

Let S_a be the vector space spanned by the matrices a^i_{2hk}, a^j_{3hk} and c^l_{1hk}, where $1 \le h,k \le n$, $\delta_k - \delta_h < i \le \delta_k$, $\quad q_h < l \le q_k - 1$, and the index j vary as follows:
if $\delta_h > 0$, $\ p_h - q_h < j \le \delta_k - q_h$, and $\max(0, p_h - \delta_k) + \delta_k - q_h < j \le \min(q_h, q_k - 1) + \delta_k - q_h$;
if $\delta_h = 0$, $\ 0 < j \le \min(q_k - 1, p_k - q_h - 1, \delta_k) + \delta_k - q_h$.

Theorem 5.8 (Second Miniversal Deformation) *Let $a_c \in \mathcal{M}^*$ be an observable marked matrix in canonical form of type (q,p) as in (5.1). Then, a miniversal deformation of $a_c \in \mathcal{M}^*$ is given by the subvariety parameterized by $\tilde{a}_c + S_{a_c}$. More explicitly, it is given by the set of matrices*

$$a_c + \left(\begin{array}{cc} 0 & X_3' \\ 0 & X_2' \\ \hline Y_1' & 0 \end{array}\right),$$

where, with the notation in (4.3), $Y_1' = Y_1^{OBS}$, $X_2' = X_2^{END}$ and X_3' is a subfamily of X_3^{FRW}.

Proof.

By construction, the set of matrices $\{a^i_{2hk}, a^j_{3hk}, c^l_{1hk}\}_{h,k,i,j,l}$ is linearly independent, and therefore the dimension of the subspace S_a spanned by them is the dimension of the orthogonal of $T_{\tilde{a}}\mathcal{O}_{\tilde{a}}$, in accordance with corollary (6.1). We will see that S_a is a supplementary subspace of $T_{\tilde{a}}\mathcal{O}_{\tilde{a}}$ by proving that its intersection is the null space. In order to do so, we will prove that for every non null vector of S_a, there is a vector of $(T_{\tilde{a}}\mathcal{O}_{\tilde{a}})^{\perp}$ such that their product is not zero.

Notice that if $\tilde{x} = (X_1, X_2, X_3, Y_1, Y_2, Z) \in \mathbb{C}^{\sigma}$, we then have $< \tilde{x}, a^i_{2hk} >= (X_{2hk})_{1,i}$, $< \tilde{x}, a^j_{3hk} >= (X_{3hk})_{1,j}$, $< \tilde{x}, c^l_{1hk} >= (Y_{1hk})_l$. Now let $v = \sum\limits_{h,k,i} x^i_{2hk} a^i_{2hk} + \sum\limits_{h,k,j} x^j_{3hk} a^j_{3hk} + \sum\limits_{h,k,l} y^l_{1hk} c^l_{1hk}$, where $x^i_{2hk}, x^j_{3hk}, y^l_{1hk} \in \mathbb{C}$, be a vector of S_a. We consider the vector $\tilde{x} = (X_1, X_2, X_3, Y_1, Y_2, Z) \in (T_{\tilde{a}}\mathcal{O}_{\tilde{a}})^{\perp}$ defined by $(X_{2hk})_{1,i} = x^i_{2hk}$,$(X_{3hk})_j = x^j_{3hk}$,$(Y_{1hk})_l = y^l_{1hk}$, where the indices vary as in (5.7). Then, $< v, \tilde{x} >= \sum\limits_{h,k,i} |x^i_{2hk}|^2 + \sum\limits_{h,k,j} |x^j_{3hk}|^2 + \sum\limits_{h,k,l} |y^l_{1hk}|^2$, and this implies that $< v, \tilde{x} >= 0$ if and only if $v = 0$. ■

Example 5.9 The new miniversal deformation in example (5.6) is

$$\left[\begin{array}{ccc|ccc|cc|c||c|cccc|cc}
 & & & & & & & & & 1 & & t_5 & & & & \\
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\
\hline
 & & & & & & & & & & t_6 & t_7 & & & & \\
 & & & 1 & & & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & & & \\
\hline
 & & & & & & & & & & & & & 1 & & \\
 & & & & & & 1 & & & & & & & & & \\
\hline
 & & & & & & & & & & & & t_3 & t_4 & & 1 \\
\hline\hline
 & & & & & & & & & * & & & & * & & * \\
\hline
 & & & & & & & & & * & * & * & * & * & * & * \\
 & & & & & & & & & & 1 & & & & & \\
 & & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & & & 1 & & & \\
\hline
 & & & & & & & & & * & & & * & * & * & * \\
 & & & & & & & & & & & & & & 1 & \\
\hline\hline
 & & 1 & & & & & & & & & & & & & \\
\hline
 & & & & & 1 & & & & & & & & & & \\
\hline
 & & & & & & & 1 & & & & & & & & \\
\hline
 & t_1 & & & t_2 & & & & 1 & & & & & & &
\end{array}\right]$$

6 Applications

6.1 Dimension of the orbit

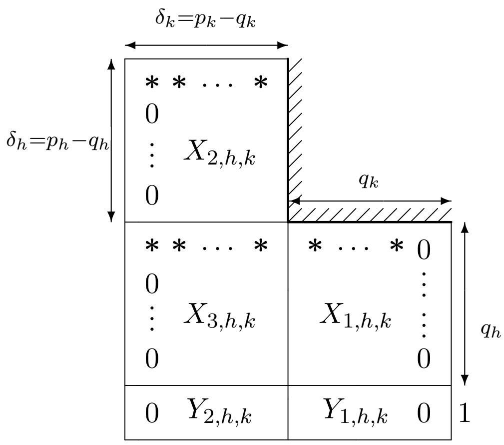
As an application of theorem (5.5), we obtain the dimension of $\mathcal{O}_{a_c}$.

Corollary 6.1 *Let $a_c \in \mathcal{M}^*$ be an observable marked matrix in canonical form of type (q, p) as in (5.1). Then, the codimension of its orbit is*

$$
\begin{aligned}
\operatorname{codim}\,\mathcal{O}_{a_c} \quad &= \sum_{\substack{1\le h,k\le n\\ \delta_h\cdot\delta_k>0}} \min(\delta_h,\delta_k) + \sum_{\substack{1\le h,k\le n\\ \delta_h\cdot\delta_k>0}} \max(0,\min(q_h,q_k-1)-\max(0,p_h-\delta_k)) + \\
&+ \sum_{\substack{1\le h,k\le n\\ \delta_h\cdot\delta_k>0}} \max(0,\delta_k-p_h) + \sum_{1\le h,k\le n} \max(0,q_k-q_h-1)\ + \\
&+ \sum_{\substack{1\le h,k\le n\\ \delta_h=0}} \min(q_k-1,p_k-q_h-1,\delta_k) + \sum_{\substack{1\le h,k\le n\\ \delta_h=0}} \max(0,\delta_k-q_h).
\end{aligned}
$$

Proof.

To count how many freedom degrees the miniversal deformation has, we study the number of parameters appearing in the solution $x = \begin{pmatrix} X_1 & X_3 \\ 0 & X_2 \\ \hline Y_1 & Y_2 \end{pmatrix}$. The following figure is useful because it shows all the nullity and constancy conditions of the diagonals appearing in theorem (5.5):



We denote by $*$ the origin of the diagonals that can be different from zero. The remaining conditions of theorem (5.5) and the sizes of the blocks will allow us to obtain the actual ones.

We distinguish three different cases depending on the number of blocks in the above figure:

(I) $\delta_h, \delta_k > 0$

In this case, all the blocks of the solution appear as the above figure shows. It can be seen that there are four types of parameters:

(a) Those beginning and finishing in $X_{2,h,k}$.
(b) Those beginning in $X_{2,h,k}$ and finishing in $Y_{1,h,k}$.
(c) Those beginning in $X_{2,h,k}$ and finishing in $Y_{2,h,k}$.
(d) Those beginning in $X_{1,h,k}$ and finishing in $Y_{1,h,k}$.

Indeed, $(X_{2,h,k})_{1,i} = (X_{2,h,k})_{1-i+\delta_k,\delta_k} \quad 1 \le i \le \delta_k$. We distinguish the cases:

(a) If $\delta_k - \delta_h < i \le \delta_k$.
(b) If $0 < i \le \delta_k, \delta_k < i + p_h < p_k, \quad (X_{2,h,k})_{1,i} = (Y_{1,h,k})_{i+p_h}$ and $i + \delta_h \le \delta_k$ because

$(Y_{1,h,k})_{q_k} = 0$.
(c) If $p_h < 1 - i + \delta_k$, $\quad (X_{2,h,k})_{1,i} = (Y_{2,h,k})_{i+p_h}$.
(d) $(X_{1,h,k})_{1,j} = (Y_{1,h,k})_{j+q_h} \qquad 1 \leq j \leq q_k$, $\quad$ and $j + q_h < q_k$ because $(Y_{1,h,k})_{q_k} = 0$.
In summary, and taking as reference the elements of the first row of $X_{2,h,k}$ in case (a) and the elements of the blocks of Y in the other ones, we have the following parameters:
(a) $(X_{2,h,k})_{1,i}$ for $\delta_k - \delta_h < i \leq \delta_k$.
(b) $(Y_{1,h,k})_j$ for $\max(0, p_h - \delta_k) < j \leq \min(q_h, q_k - 1)$.
(c) $(Y_{2,h,k})_j$ for $p_h < j \leq \delta_k$.
(d) $(Y_{1,h,k})_i$ for $q_h < i \leq q_k - 1$.
Adding up these four types of parameters, classified according to whether they finish in $X_{2,h,k}$, $Y_{2,h,k}$ or $Y_{1,h,k}$, respectively, for the case $\delta_h, \delta_k > 0$ we obtain the following total number of freedom degrees:

$$\min(\delta_h, \delta_k) + \max(0, \min(q_h, q_k - 1) - \max(0, p_h - \delta_k)) + \max(0, \delta_k - p_h) + \max(0, q_k - q_h - 1)\,.$$

(II) $\delta_k = 0$
In this case, we only have the blocks $X_{1,h,k}$ and $Y_{1,h,k}$. Repeating the reasoning of (I), if $\delta_k = 0$, there only exist the parameters beginning in $X_{1,h,k}$, that is, those given by (d), and the number of parameters is

$$\max(0, q_k - q_h - 1)\,.$$

(III) $\delta_h = 0, \delta_k > 0$
In this case, we only have the blocks $X_{1,h,k}$, $X_{3,h,k}$, $Y_{1,h,k}$ and $Y_{2,h,k}$. It can be seen that there are three types of parameters:
(d) Those beginning in $X_{1,h,k}$ and finishing in $Y_{1,h,k}$.
(e) Those beginning in $X_{3,h,k}$ and finishing in $Y_{1,h,k}$.
(f) Those beginning in $X_{3,h,k}$ and finishing in $Y_{2,h,k}$.
Adding up these three types of parameters, for the case $\delta_h = 0, \delta_k > 0$ we obtain the following total number of freedom degrees:

$$\max(0, q_k - q_h - 1) + \min(q_k - 1, p_k - q_h - 1, \delta_k) + \max(0, \delta_k - q_h)\,.$$

Adding up all the preceding cases and grouping them according to the block of the reference parameter, we have the formula of the corollary. ■

Example 6.2 For the observable marked matrix in (5.2), the addends in (6.1) give, respectively, 15,2,1,2,1 and 1, which correspond to the parameters (*), (t_4,t_5), (t_3), (t_1,t_2), (t_7) and (t_6).

6.2 Structural stability

Now we study the observable marked matrices for which all the small perturbations do not change their equivalence class. We will see that there are no non-trivial pairs of this type.

Corollary 6.3 *The observable marked matrices $a \in \mathcal{M}^*$ structurally stable with regard to the equivalence relation in (3.7) are only those of the trivial case:*

(i) $d = N$,
(ii) $|p_i - p_j| \leq 1$.

Proof.

Let $a \in \mathcal{M}^*$ be an observable marked matrix in canonical form of type (q,p). Then, it will be structurally stable if and only if its miniversal deformation is null. Or, equivalently, if codim $\mathcal{O}_a = 0$. This means that each term in (6.1) must be 0. From the first one, it follows $\delta_i = 0$, for all $i \leq n$, and then $d = N$. Now the fourth term gives us $|p_k - p_h| \leq 1 \quad \text{for} \quad 1 \leq h,k \leq m$.

Conversely, all the terms are null if (i)-(ii) hold. ■

Notice that condition (i) implies $\mathcal{M}^* = M_{N+n,N}(\mathbb{C})$, that is, $a \in \mathcal{M}^*$ is a full rank observable pair. Then, condition (ii) is exactly that obtained in [12] for this particular case.

6.3 Bifurcation diagrams

Let us study the effect of the parameters $X_3^{'}$ and $Y_1^{'}$ in (5.8). It is clear that the Jordan form of A_2 is perturbed by $X_2^{'}$ just as in Arnold's works.

With regard to the effects of the restriction (C_1, A_1) on the BK-indices, it is also clear that they only depend on $Y_1^{'}$ and are just those of the ordinary perturbations of an observable pair. So the initial BK-indices $q_1, \ldots, q_n$ will change into new indices $q_1^{'}, \ldots, q_n^{'}$ such that they are *majorized* by the initial ones in the following sense (see [19]):
$q_1^{'} \leq q_1$; $q_1^{'} + q_2^{'} \leq q_1 + q_2$; $\cdots$; $q_1^{'} + \cdots + q_{n-1}^{'} \leq q_1 + \cdots + q_{n-1}$; $q_1^{'} + \cdots + q_n^{'} = q_1 + \cdots + q_n$.

That is, the BK-indices of the restriction will be balanced until the maximal difference between them is 1 (or 0).

In (5.9), for instance, if some of the parameters t_1, t_2 in $Y_1^{'}$ are non zero, then $(q_1^{'}, q_2^{'}, q_3^{'}, q_4^{'}) = (3,2,2,2)$.

Note that they are the only BK-indices compatible with the above majorization relations and are structurally stable because the maximal difference between them is 1.

Similarly, the BK-indices of (C, A) will be perturbed into majorized ones as above. For instance, in (5.9) the only BK-indices majorized by the initial ones $(6,4,3,3)$ are $(5,5,3,3)$, $(5,4,4,3)$ and $(4,4,4,4)$. In fact, the bifurcation diagram in $X_3^{'}$ is
$(6,4,3,3)$ if $t_3 = t_4 = t_5 = t_6 = t_7 = 0$,
$(5,5,3,3)$ if $t_5 \neq 0, \quad t_3 = t_4 = t_6 = t_7 = 0$,
$(5,4,4,3)$ if $t_4 t_6 - t_3 t_7 = 0$, some of them being non zero,

$(4,4,4,4) \quad \text{if} \quad t_4t_6 - t_3t_7 \neq 0.$

Notice that the above bifurcation diagram is part of that of the miniversal deformation in (4.7) when one prescribes the 3 parameters of X_3^{FRW} appearing in (5.2).

6.4 BK-deformation and S-deformation

The S-equivalence in (2.6) is, in general, finer than the ordinary block similarity or BK-equivalence because the condition of (C, A) and (C', A') being BK-equivalent does not imply the existence of P verifying not only (i) but also (ii). In fact, (2.3) shows that a necessary condition for the S-equivalence (but not for the BK-equivalence) is that the restriction of both pairs to S is block-similar. However, neither this condition is sufficient. For example, taking

$$S = \mathbb{C} \times \{(0,0)\} \subset \mathbb{C}^3, \qquad \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 0\ 1\ 0 \\ 0\ 0\ 0 \\ 0\ 0\ 0 \\ \hline 1\ 0\ 0 \end{pmatrix}, \quad \begin{pmatrix} A' \\ C' \end{pmatrix} = \begin{pmatrix} 0\ 1\ 0 \\ 0\ 0\ 0 \\ 0\ 1\ 0 \\ \hline 1\ 0\ 0 \end{pmatrix}$$

one has that both pairs are block-similar, and so are their restrictions to S. Nevertheless, a straightforward computation shows that no P verifies (i)-(ii).

Therefore, we have

$$\mathcal{O}_a \subset \mathcal{M}_r \cap \mathcal{O}_{BK}(a),$$

where the latter means the BK-equivalence class of a, *but the converse inclusion does not hold in general.*

In the particular case of a BK-matrix a_{BK} and $S = \mathbb{C}^d \times \{0\}$ as in (4.6), in [12] one obtains a BK-miniversal deformation which is contained in $\mathcal{M}_r$. Hence, $\mathcal{M}_r$ is transverse to $\mathcal{O}_{BK}(a_{BK})$ at a_{BK}; therefore, in a neighborhood of it, the above intersection $\mathcal{M}_r \cap \mathcal{O}_{BK}(a_{BK})$ is a differentiable manifold. In addition, its dimension is just that of $\mathcal{O}_{a_{BK}}$ because (4.6) shows that the BK-miniversal deformation in [12] is also an S-miniversal deformation. Thus, *the converse is true in a neighborhood of a_{BK}.*

However, notice that, although the same miniversal deformation of a_{BK} is valid with regard to both equivalence relations, *the corresponding bifurcation diagrams can be different.* For instance, for observable deformations (in fact, a generic case) the BK-bifurcation diagram involves only the BK-indices of the pair whereas the S-one must consider other invariants, such as the BK-indices of the restriction to S.

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