Pencils with Prescribed Constant Subpencils *

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Abstract

The results in this paper are within the scope of the matrix pencil completion problems. Given an arbitrary matrix pencil, we obtain necessary and sufficient conditions for the existence of a pencil strictly equivalent to it with a prescribed constant subpencil, in terms of very simplified conditions and for algebraically closed fields.

1 Introduction

Let $\mathbb{F}$ be an algebraically closed field. Let $A, B \in \mathbb{F}^{m\times n}$. Let $C \in \mathbb{F}^{p\times q}$, with $p \leq m,\ q \leq n$. Let x, y be indeterminate.

Our purpose is to solve the following problem:

Problem 1.1 *Under what conditions there exists a pencil strictly equivalent to $Ax + B$ containing the constant pencil C as a subpencil?*

Note that, if $P \in \mathbb{F}^{p\times p},\ Q \in \mathbb{F}^{q\times q}$ are nonsingular matrices such that PCQ has the form

$$\begin{bmatrix} I_\rho & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } \rho = \operatorname{rank} C, \tag{1}$$

then there exists a pencil strictly equivalent to $Ax + B$ containing C as a subpencil if and only if there exist a pencil strictly equivalent to $Ax + B$

*Partially supported by Proyecto de Investigación MCM2004-06389-C02-01 and CELC

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containing PCQ as a subpencil. From now on, without loss of generality, we shall assume that C has the form (1).

Solutions to particular cases are known, with different restrictions on the underlying field. They are frequently consequences of results on matrix pencil completion problems [7]. For infinite fields, if $C = I_p$ and $Ax + B$ is regular, the result can be found in [1]. Also for regular pencils, with $\det A \neq 0$ and $p + q = m$, the solution to the problem can be found in [6] for infinite fields, and in [11, 14] for algebraically closed fields. An extension of this result to the case $p + q \leq m$ can be derived from a result of [12]. Particular attention deserves the result obtained in [3], which characterizes the completion of an arbitrary pencil to a regular one, in terms of an existence condition, for infinite fields. This result covers the solution to our problem when $Ax + B$ is regular.

To find other particular cases of matrix pencil completion problems see [2, 5]. Recently, an explicit solution to the result given in [3] has been obtained, for algebraically closed fields (see [4]).

Our result is proven when $\mathbb{F}$ is an algebraically closed field, and in terms of very simple conditions. In particular, we provide another solution to Problem 1.1 in terms of simple conditions when $Ax + B$ is regular, over algebraically closed fields.

The paper is organized as follows. Notation and preliminary results are introduced in section 2. In section 3 we obtain necessary conditions. Solutions to different particular cases are obtained in subsequent sections. In section 4, for the case where the prescribed subpencil C is row complete ($p = m$). In section 5, for rank $C = 0$. When $Ax + B$ is regular, in section 6. For pencils having only column minimal indices in section 7, and for pencils having a regular part and column minimal indices, in section 8. Finally, we obtain the solution to the general case in section 9.

2 Notation and preliminary results

Let $d_c(Ax + B)$ be the dimension of the $\mathbb{F}$-subspace of $\mathbb{F}[x]^{m\times 1}$ spanned by the columns of $Ax + B$. Analogously, let $d_r(Ax + B)$ be the dimension of the $\mathbb{F}$-subspace of $\mathbb{F}[x]^{1\times n}$ spanned by the rows of $Ax + B$. Note that the numbers $d_c(Ax + B)$, $d_r(Ax + B)$ are invariant under strict equivalence and are easy to compute when the pencil is in Kronecker canonical form. Hence, the following property is satisfied.

Lemma 2.1 *Let $Ax + B$ and $A'x + B'$ be strictly equivalent pencils. Then $d_c(Ax + B) = d_c(A'x + B')$ and $d_r(Ax + B) = d_r(A'x + B')$.*

The nonzero columns of the Kronecker canonical form of $Ax + B$ are linearly independent as vectors of the $\mathbb{F}$-space $\mathbb{F}[x]^{m\times 1}$. As a consequence, we have the following result.

Lemma 2.2 *Given a pencil $Ax + B$, $d_c(Ax + B)$ is equal to rank $(Ax + B)$ plus the number of nonzero column minimal indices of $Ax+B$. Analogously, $d_r(Ax+B)$ is equal to rank $(Ax+B)$ plus the number of nonzero row minimal indices of $Ax + B$.*

We use the following notation:

$d_c = d_c(Ax + B)$, $d_r = d_r(Ax + B)$.
u is number of nonzero column minimal indices of $Ax + B$. It will be represented by $u(Ax + B)$ if necessary.
$c_1 \leq c_2 \leq \ldots \leq c_u$ are the nonzero column minimal indices of $Ax + B$,
v ($v(Ax + B)$ if necessary) is the number of nonzero row minimal indices of $Ax + B$, and $r_1 \leq r_2 \leq \ldots \leq r_v$ the nonzero row minimal indices of $Ax + B$.
$\delta_c = \min\{q - \rho, n - d_c\}$, $\delta_r = \min\{p - \rho, m - d_r\}$.
$j_c = j_c(Ax + B)$ is the largest nonnegative integer j such that $Ax + B$ has j nonzero column minimal indices $c_1 \leq \ldots \leq c_j$ whose sum does not exceed $m - p$ and $c_1 + \ldots + c_j + j \leq q - \rho - \delta_c$,
$j_r = j_r(Ax + B)$, the largest nonnegative integer j such that $Ax + B$ has j nonzero row minimal indices $r_1 \leq \ldots \leq r_j$ whose sum does not exceed $n - q$ and $r_1 + \ldots + r_j + j \leq p - \rho - \delta_r$.
t ($t(Ax + B)$ if necessary) is the number of infinite elementary divisors of $Ax + B$, and
t_1 ($t_1(Ax + B)$ if necessary) the number of infinite elementary divisors of $Ax + B$ of degree greater than one,
$1 \leq k_1 \leq k_2 \leq \ldots \leq k_t$ are the degrees of the infinite elementary divisors of $Ax + B$.
Given two polynomials $\alpha, \beta \in \mathbb{F}[x]$, $\alpha \mid \beta$ means that α divides β.
Whenever a sequence of polynomials satisfy $\tau_1 \mid \ldots \mid \tau_n$, we will assume that $\tau_i = 1$ for $i < 1$ and $\tau_i = 0$ for $i > n$.

$\alpha_1 \mid \ldots \mid \alpha_\rho$ denote the homogeneous invariant factors of C (notice that $\alpha_i = y,\ i = 1, \ldots, \rho$) and $\gamma_1 \mid \ldots \mid \gamma_\omega$ with $w = \text{rank } (Ax + B)$, the homogeneous invariant factors of $Ax + B$.
$l = \max\{i, t_1\}$ where i denotes the amount of nontrivial invariant factors of $Ax + B$.
$d(\cdot)$ stands for 'the degree of'.

Lemma 2.3 *Let $H = [G\ *] \in \mathbb{F}[x]^{m\times h}$ be a matrix pencil with $G \in \mathbb{F}[x]^{m\times g}$. Let $c'_1 \leq \ldots \leq c'_{u'}$ and $c_1 \leq \ldots \leq c_u$ be the column minimal indices of H and G, respectively. Then $u' \geq u$ and $c'_i \leq c_i,\ i = 1, 2, \ldots, u$.*

Proof: Taking into account that $u' = h - \text{rank } H$ and $u = g - \text{rank } G$, we conclude that $u' \geq u$.

Let $X_1, \ldots, X_{u'}$ and $Y_1, \ldots, Y_u$ be fundamental series of solutions of $HX = 0$ and $GY = 0$, respectively. For every $i \in \{1, \ldots, u\}$, let

$$Z_i = \begin{bmatrix} Y_i \\ 0 \end{bmatrix} \in \mathbb{F}[x]^{h\times 1}.$$

Clearly, $Z_1, \ldots, Z_u$ are solutions of $HX = 0$.

By definition of a fundamental series of solutions, $c'_1 = \deg X_1 \leq \deg Z_1 = \deg Y_1 = c_1$. Let $i \in \{1, \ldots, u-1\}$. As $Z_1, \ldots, Z_{i+1}$ are linearly independent, there exists $j \in \{1, \ldots, i+1\}$ such that Z_j is not a linear combination (as a vector in the $\mathbb{F}(x)$-space $\mathbb{F}(x)^{h\times 1}$) of $X_1, \ldots, X_i$. Then $X_1, \ldots, X_i, Z_j$ are linearly independent. By definition of a fundamental series of solutions

$$c'_{i+1} = \deg X_{i+1} \leq \deg Z_j = \deg Y_j \leq \deg Y_{i+1} = c_{i+1},$$

and the conclusion follows. ■

Lemma 2.4 *Let $H = [G\ *] \in \mathbb{F}[x]^{m\times h}$ be a matrix pencil with $G \in \mathbb{F}[x]^{m\times g}$. Let $\bar{c}'_1 \leq \ldots \leq \bar{c}'_{\bar{u}'}$ and $\bar{c}_1 \leq \ldots \leq \bar{c}_{\bar{u}}$ be the nonzero column minimal indices of H and G, respectively. Then $\bar{u}' \geq \bar{u}$ and $\bar{c}'_i \leq \bar{c}_i,\ i = 1, 2, \ldots, \bar{u}$.*

Proof: Let

$$X = \begin{bmatrix} Y & R \\ 0 & W \end{bmatrix} \in \mathbb{F}[x]^{h\times h},$$

be a non singular matrix, $Y \in \mathbb{F}^{g\times g}$, such that

$$HX = [\ 0\ G_0 \mid H_0\ 0\ ],$$

where the columns of $[G_0 \ H_0]$ are linearly independent as vectors in the $\mathbb{F}$-space $\mathbb{F}[x]^{m\times 1}$. Then $\bar{c}'_1, \ldots, \bar{c}'_{\bar{u}'}$ and $\bar{c}_1, \ldots, \bar{c}_{\bar{u}}$ are the column minimal indices of $[G_0 \ H_0]$ and G_0, respectively. The conclusion follows from the former Lemma. ■

As a consequence, we obtain the following result.

Lemma 2.5 *Assume that $Ax + B$ is strictly equivalent to a pencil of the form*

$$\begin{bmatrix} 0_{p\times q} & * \\ Cx + D & * \end{bmatrix} \in \mathbb{F}[x]^{m\times n}.$$

If $Cx + D$ has j nonzero column minimal indices, then $Ax + B$ has, at least, j nonzero column minimal indices whose addition does not exceed $m - p$.

Notice that if α is the addition of the j column minimal indices of $Ax+B$ in the former lemma, then $\alpha + j \leq q$.

An important result concerning completion of matrices was given by E.M. de Sá [10] and R.C. Thompson [13] independently, and is the following

Theorem 2.6 [10], [13] *Let $A(x) \in \mathbb{F}[x]^{p\times q}$ and $B(x) \in \mathbb{F}[x]^{m\times n}$, $p \leq m$, $q \leq n$, be polynomial matrices, $\alpha_1 \mid \ldots \mid \alpha_p$ and $\gamma_1 \mid \ldots \mid \gamma_m$ their invariant factors, respectively. Then, there exist matrices $X(x) \in \mathbb{F}[x]^{p\times(n-q)}$, $Y(x) \in \mathbb{F}[x]^{(m-p)\times q}$, $Z(x) \in \mathbb{F}[x]^{(m-p)\times(n-q)}$ such that $B(x)$ is strictly equivalent to*

$$\begin{bmatrix} A(x) & X(x) \\ Y(x) & Z(x) \end{bmatrix}$$

if and only if

$$\gamma_i \mid \alpha_i \mid \gamma_{i+m-p+n-q}, \quad i = 1, \ldots, p.$$

In both papers [10, 13] the following result on constant matrices was also obtained.

Theorem 2.7 [10], [13] *Let $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{m\times m}$ be matrices, $\alpha_1 \mid \ldots \mid \alpha_p$, $\gamma_1 \mid \ldots \mid \gamma_m$ their invariant factors, and $p \leq m$. Then there exist matrices $X \in \mathbb{F}^{p\times(m-p)}$, $Y \in \mathbb{F}^{(m-p)\times p}$, $Z \in \mathbb{F}^{(m-p)\times(m-p)}$ such that B is similar to*

$$\begin{bmatrix} A & X \\ Y & Z \end{bmatrix}$$

if and only if

$$\gamma_i \mid \alpha_i \mid \gamma_{i+2(m-p)}, \quad i = 1, \ldots, p.$$

We include below some results showing solutions to Problem 1.1, which will be mentioned or used later. A particular case of the next Theorem shows the solution to Problem 1.1 when the pencil $Ax + B$ and the prescribed subpencil are both regular and the underlying field is infinite.

Theorem 2.8 [1] *Let $\mathbb{F}$ be an infinite field. Let $Gx+H \in \mathbb{F}[x]^{p\times p}$, $Ax+B \in \mathbb{F}[x]^{m\times m}$ be regular pencils, $p \leq m$. Let $\alpha_1 \mid \ldots \mid \alpha_p$, $\gamma_1 \mid \ldots \mid \gamma_m$ be their homogeneous invariant factors, respectively. Then there exists a pencil strictly equivalent to $Ax + B$, containing $Gx + H$ as a subpencil if and only if*

$$\gamma_i \mid \alpha_i \mid \gamma_{i+2(m-p)}, \ i = 1, \ldots, p.$$

The following result contains the solution to Problem 1.1 for regular pencils $Ax + B$ with $\det A \neq 0$ and $p + q = m$, for algebraically closed fields. It can be obtained from the results in [11, 14] (it can also be found in [6] for infinite fields).

Theorem 2.9 [11, 14] *Let $\mathbb{F}$ be an algebraically closed field. Let $Ax + B \in \mathbb{F}[x]^{m\times m}$ be regular,* $\det A \neq 0$, *$p + q = m$. Let $\gamma_1 \mid \ldots \mid \gamma_n$ be the invariant factors of $Ax + B$. Then there exists solution to Problem 1.1 if and only if*

$$\gamma_\rho \mid 1.$$

As a consequence we obtain a generalization of it to the case $p + q \leq m$. The result can be derived as a particular case from a result of [12].

Lemma 2.10 *Let $\mathbb{F}$ be an algebraically closed field. Let $Ax + B$ be regular,* $\det A \neq 0$. *Let $\gamma_1 \mid \ldots \mid \gamma_m$ be the invariant factors of $Ax + B$. Then there exists solution to Problem 1.1 if and only if*

$$p + q \leq m, \tag{2}$$

$$\rho \leq m - i, \tag{3}$$

where i is the amount of nontrivial invariant factors of $Ax + B$.

Proof: Assume that $Ax + B$ is strictly equivalent to a pencil of the form

$$\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix},$$

with rank $C = \rho$. Then rank $A \leq m - p + m - q$, from where we obtain condition (2). By definition of invariant factor, $\gamma_\rho = 1$ and condition (3) is also satisfied.

Conversely. Let $\tilde{C}$ be a matrix of the form

$$\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{F}^{p_1 \times q_1},$$

such that $p \leq p_1,\ q \leq q_1$ and $p_1 + q_1 = m$. If condition (3) is satisfied, by Theorem 2.9 the pencil $Ax + B$ is strictly equivalent to a pencil containing $\tilde{C}$ as a subpencil. In particular, it contains the matrix C. ■

3 Necessary conditions

In this section we obtain necessary conditions which follow if Problem 1.1 has solution. Although in some cases, which will be studied later (in sections 4 and 5), the stated necessary conditions can be obtained straightforward, we will derive them from the result obtained in this section.

Theorem 3.1 *Let $Ax + B$ be a pencil. Assume that $d_c = n, d_r = m$. Let $Ax + B$ be strictly equivalent to a pencil of the form*

$$\begin{bmatrix} C & * \\ * & * \end{bmatrix},$$

with rank $C = \rho$. Then the following conditions are satisfied

$$\text{rank } A \leq m - p + n - q, \tag{4}$$

$$\text{rank } (Ax + B) \geq p + q - \rho - j, \tag{5}$$

where

$$j = \max\{k \in \{0, 1, \ldots, j_c + j_r\} : \rho \leq \text{rank } (Ax + B) - l - (d_1 + \ldots + d_k)\},$$

where $l = \max\{i, t_1\}$, and the integers $\{d_1, \ldots, d_{j_c + j_r}\}$ are an increasing reordering of $\{c_1, \ldots, c_{j_c}, r_1, \ldots, r_{j_r}\}$, with the convention $d_1 + \ldots + d_k = 0$ if $k = 0$.

Proof: Suppose that

$$Ax+B \overset{st}{\sim} \begin{bmatrix} I_\rho & 0 & * \\ 0 & 0 & G' \\ * & G & * \end{bmatrix},$$

with $G \in \mathbb{F}^{(m-p)\times(q-\rho)}$, $G' \in \mathbb{F}^{(p-\rho)\times(n-q)}$. As $d_c = n, d_r = m$, $\delta_c = 0$ and $\delta_r = 0$. It is immediate that condition (4) is satisfied. By Lemma 2.5 the nonzero column minimal indices of G are also column minimal indices of $Ax+B$. Denote them by $c_{i_1}, \ldots, c_{i_k}$. Notice that $c_{i_1} + \ldots + c_{i_k} \leq m-p$ and $c_{i_1} + \ldots + c_{i_k} + k \leq q - \rho$. According to the definition of j_c, we have that $k \leq j_c$. In the same way, the row minimal indices $r_{i_1}, \ldots, r_{i_{k'}}$ of G' satisfy that $r_{i_1} + \ldots + r_{i_{k'}} \leq n-q$ and $r_{i_1} + \ldots + r_{i_{k'}} + k' \leq p - \rho$; hence, $k' \leq j_r$. As

$$\text{rank } (Ax+B) \geq \rho + c_{i_1} + \ldots + c_{i_k} + r_{i_1}, \ldots, r_{i_{k'}} + l,$$

it follows that

$$\rho \leq \text{rank } (Ax+B) - l - (c_{i_1} + \ldots + c_{i_k} + r_{i_1}, \ldots, r_{i_{k'}}) \leq$$

$$\leq \text{rank } (Ax+B) - l - (c_1 + \ldots + c_k + r_1, \ldots, r_{k'}),$$

and we conclude that $k + k' \leq j$. Moreover,

$$\begin{bmatrix} I_\rho & 0 & * \\ 0 & 0 & G' \\ * & G & * \end{bmatrix} \overset{st}{\sim} \left[\begin{array}{ccc|c|c} I_\rho & 0 & 0 & * & * \\ 0 & 0 & 0 & G_1' & 0 \\ 0 & 0 & 0 & 0 & G_2' \\ \hline * & G_1 & 0 & * & * \\ \hline * & 0 & G_2 & * & * \end{array}\right],$$

where G_1 has the column minimal indices $c_{i_1}, \ldots, c_{i_k}$ and G_1' the row minimal indices $r_{i_1}, \ldots, r_{i_{k'}}$ as Kronecker invariants. Then,

$$\text{rank } (Ax+B) \geq \rho + \text{rank } G + \text{rank } G' = \rho + q - \rho - k + p - \rho - k' \geq p + q - \rho - j,$$

and condition (5) holds. ∎

Let us obtain necessary conditions for the general case.

Theorem 3.2 *Let $Ax + B$ be a pencil strictly equivalent to a pencil of the form*

$$\begin{bmatrix} C & * \\ * & * \end{bmatrix},$$

with rank $C = \rho$. Then the following conditions are satisfied

$$\text{rank } A \leq d_r - (p - \delta_r) + d_c - (q - \delta_c), \tag{6}$$

$$\text{rank } (Ax + B) \geq p - \delta_r + q - \delta_c - \rho - j, \tag{7}$$

where j is defined as in the former theorem.

Proof: Assume that $Ax + B$ is strictly equivalent to a pencil of the form

$$\begin{bmatrix} I_\rho & 0 & * \\ 0 & 0 & G' \\ * & G & * \end{bmatrix},$$

with $G \in \mathbb{F}^{(m-p)\times(q-\rho)}$, $G' \in \mathbb{F}^{(p-\rho)\times(n-q)}$. There may be some zero columns in the block G and some zero rows in the block G'. In addition, there may be some other zero columns or rows in the pencil. Let δ be the amount of zero columns of block G, and δ' the amount of zero rows of G'. Then, $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|cc|c|c} I_\rho & 0 & 0 & * & 0 \\ \hline 0 & 0 & 0 & G_1' & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline * & G_1 & 0 & * & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

with $G_1 \in \mathbb{F}^{(d_r-p+\delta'))\times(q-\rho-\delta)}$, $G' \in \mathbb{F}^{(p-\rho-\delta')\times(d_c-q+\delta')}$, and $d_c(G_1) = q - \rho - \delta$ and $d_r(G_1') = p - \rho - \delta'$. Notice that $\delta \leq \delta_c$, $\delta' \leq \delta_r$.

Through column and row permutation we may achieve that the δ-width zero column expands to a δ_c-width zero column, and the δ'-width zero row to a δ_r-width zero row, the pencil being then strictly equivalent to

$$\left[\begin{array}{c|cc|c|ccc} I_\rho & 0 & 0 & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & G_2' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline * & G_2 & 0 & * & * & 0 & 0 \\ \hline * & 0 & 0 & * & 0 & G_3 & 0 \\ 0 & 0 & 0 & 0 & G_3' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

with $G_1 = \mathrm{Diag}\,(G_2, G_3)$, $G'_1 = \mathrm{Diag}\,(G'_2, G'_3)$ for some matrices G_2, G_3, G'_2, G'_3. Put $p' = p - \delta_r$, $q' = q - \delta_c$. The former pencil is strictly equivalent to

$$\left[\begin{array}{c|c} A'x+B' & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{ccccc|cc} I_\rho & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & G'_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G'_3 & 0 & 0 \\ * & G_2 & 0 & * & * & 0 & 0 \\ * & 0 & G_3 & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Then $A'x + B' \in \mathbb{F}^{d_r \times d_c}$ contains a $p' \times q'$ constant subpencil of rank ρ, $d_r(A'x + B') = d_r$, $d_c(A'x + B') = d_c$.

Let us calculate $j' = j(A'x + B')$. We have that

$$d_r - p' = d_r - (p - \delta_r) = m - p - (m - d_r) + \delta_r \le m - p,$$

$$p' - \rho = p - \delta_r - \rho,$$

therefore, $j'_r = j_r(A'x + B') \le j_r$. Analogously, $j'_c = j_c(A'x + B') \le j_c$. Then $j' \le j$. By the former theorem,

$$\mathrm{rank}\ A = \mathrm{rank}\ A' \le d_r - p' + d_c - q' = d_r - (p - \delta_r) + d_c - (q - \delta_c),$$

$$\mathrm{rank}\ (Ax + B) = \mathrm{rank}\ (A'x + B') \ge p' + q' - \rho - j' \ge p - \delta_r + q - \delta_c - \rho - j,$$

what are the desired conditions (6) and (7). ■

Remark: Assume that $\alpha_1 \mid \dots \mid \alpha_\rho$ are the invariant factors of C and $\gamma_1 \mid \dots \mid \gamma_{\mathrm{rank}\ (Ax+B)}$ the invariant factors of $Ax + B$. According to Theorem 2.6, if there is solution to Problem 1.1 the interlacing conditions

$$\gamma_i \mid \alpha_i \mid \gamma_{i+(m-p)+(n-q)}, \ i = 1, \dots, \rho. \tag{8}$$

must be satisfied. Let us see that these conditions are equivalent to the following conditions (i) to (iii):

(i) $\mathrm{rank}\ A \le m - p + n - q$.

(ii) $\mathrm{rank}\ (Ax + B) \le m - p + n - q + \rho$.

(iii) $\rho \le \mathrm{rank}\ (Ax + B) - \max\{i, t_1\}$.

It is easy to see that (8) implies that $\gamma_\rho \mid y$, what is condition (iii); $\gamma_{\rho+(m-p)+(n-q)+1} = 0$, i.e. rank $(Ax+B) \leq (m-p)+(n-q)+\rho$, what is condition (ii); $y \mid \gamma_{i+(m-p)+(n-q)}$ for $i = 1, \ldots,$ rank $(Ax+B)-(m-p+n-q)$, hence $t \geq$ rank $(Ax+B)-(m-p+n-q)$, therefore $t+m-p+n-q \geq$ rank $(Ax+B) =$ rank $A+t$, and (i) is satisfied.

Conversely. Assume that (i) to (iii) are satisfied. Condition (iii) implies that $\gamma_i \mid \alpha_i,\ i = 1, \ldots, \rho$. By (i), $t \geq$ rank $(Ax+B)-(m-p+n-q)$, hence $\alpha_i = y \mid \gamma_{i+(m-p)+(n-q)},\ i = 1, \ldots,$ rank $(Ax+B)-(m-p+n-q)$. From (ii), rank $(Ax+B)-(m-p+n-q) \leq \rho$, then $\alpha_i \mid \gamma_{i+(m-p)+(n-q)} = 0,\ i =$ rank $(Ax+B)-(m-p+n-q)+1, \ldots, \rho$. Therefore $\alpha_i \mid \gamma_{i+(m-p)+(n-q)},\ i = 1, \ldots, \rho$. As a result, the interlacing conditions (8) are satisfied.

Therefore, if Problem 1.1 has solution, conditions (i) to (iii) must be satisfied. They can be derived from conditions (6), (7) and from the definition of j, respectively.

4 The case $Ax+B$ arbitrary pencil, $p = m$

We obtain in this section the solution to the particular case of Problem 1.1 where the prescribed subpencil C is row complete.

Theorem 4.1 *Let $Ax+B \in \mathbb{F}^{m\times n}$ be an arbitrary pencil. Let* u *be the amount of its nonzero column minimal indices and* t *the amount of its infinite elementary divisors. Let $C \in \mathbb{F}^{p\times q}$, rank $C = \rho$. Assume that* p = m. *Then there exists solution to Problem 1.1 if and only if*

$$\rho \leq t+u, \tag{9}$$

$$d_c \leq n-q+\rho. \tag{10}$$

Proof: The necessity is a consequence of Theorem 3.2. Let us proof the sufficiency.

The pencil $Ax+B$ is strictly equivalent to a pencil of the form $\begin{bmatrix} A'x+B' & 0 \end{bmatrix}$, where $d_c(A'x+B') = d_c$. From condition (10) $q-\rho \leq n-d_c$, therefore it is enough to prove that $A'x+B'$ is strictly equivalent to a pencil of the form

$$\begin{bmatrix} * & I_\rho \\ * & 0 \end{bmatrix}.$$

But this is obvious form the Kronecker canonical form of $Ax+B$ and condition (9). ■

5 The case $Ax + B$ arbitrary pencil, $\rho = 0$.

From now on we will assume that $\mathbb{F}$ is algebraically closed. Whenever a result holds also for arbitrary fields, it will be appropriately pointed out.

Next Theorem shows a solution to Problem 1.1 when the prescribed submatrix is a zero matrix. In [9] the authors provided a characterization to this case in terms of different conditions. We will prove later that both conditions are equivalent.

Theorem 5.1 *Let $Ax+B \in \mathbb{F}[x]^{m\times n}$. Let $C \in \mathbb{F}^{p\times q}$, with rank $C = 0$. Then $Ax + B$ is strictly equivalent to a pencil of the form*

$$\begin{bmatrix} 0_{p\times q} & * \\ * & * \end{bmatrix},$$

if and only if

$$\text{rank } (Ax + B) \geq p - \delta_r + q - \delta_c - j, \tag{11}$$

with j defined as in Theorem 3.1.

Proof: The necessity is a consequence of Theorem 3.2. Let us proof the sufficiency.

Assume that condition (11) is satisfied. Note that by definition $j = j_r+j_c$. Remind that $\delta_r = \min\{p, m - d_r\}$, $\delta_c = \min\{q, n - d_c\}$. We analyze different cases according to the values of δ_r and δ_c.

If $\delta_r = p$ or $\delta_c = q$ the result follows trivially.

Let $\delta_r = m - d_r$ and $\delta_c = n - d_c$. We may assume that $\delta_r < p, \delta_c < q$.
If $p = m$ or $q = n$, the result follows from Theorem 4.1. Suppose then that $p < m$ and $q < n$. We analyze different cases.

<u>Case 1</u>: Let $Ax + B$ be a regular pencil. Then $\delta_r = \delta_c = 0$, $j = 0$ and rank $(Ax + B) = m \geq p + q$.

Subcase 1.1: If $\det A \neq 0$, the result follows from Theorem 2.10.

Subcase 1.2: If $\det A = 0$, we obtain the result by induction on $m + n$.
If $m = n = 2$ the result is trivial. Let $m + n > 4$.

If $k_1 = 1$ then $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|c} 0 & A'x + B' \\ \hline 1 & 0 \end{array}\right],$$

and rank $(A'x + B') = \text{rank } (Ax + B) - 1 \geq p - 1 + q - 1$. By induction hypothesis, the result follows.

If $k_1 > 1$ then $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|cccc} 0 & & A'x+B' & & \\ \hline 1 & x & 0 & \cdots & 0 \end{array}\right],$$

and rank $(A'x+B') = \text{rank } (Ax+B)-1 \geq p+q-1$. By induction hypothesis, the result follows.

<u>Case 2</u>: Let $Ax + B$ be an arbitrary pencil. Assume it has column minimal indices $(u > 0)$. We also study different subcases.

Subcase 2.1: Let $j_c = 0$. Then $c_1 > m - p$ or $c_1 + 1 > q - \delta_c$, and $j = j_r$. If $m = n = 2$ the result is trivial. Let $m + n > 4$.

If $c_1 = 1$ then $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|c|c} 0 & 0 & A'x+B' \\ \hline x & 1 & 0 \end{array}\right].$$

Notice that, if we define $p' = p-1$ and $q' = q-1$ then $\delta_r' = \delta_r(A'x+B') = \delta_r$, $\delta_c' = \delta_c(A'x+B') = \delta_c$, $j_c(A'x+B') = 0$ $(c_2+1 > q-1-\delta_c)$, $j_r - j_r(A'x+B') \leq 1$ and $j' = j(A'x + B') = j_r(A'x + B')$. We have that

$$\text{rank } (A'x + B') = \text{rank } (Ax + B) - 1 \geq p - 1 - \delta_r + q - 1 - \delta_c - (j_r - 1) \geq$$

$$\geq p' - \delta_r' + q' - \delta_c' - j',$$

and the result follows by induction hypothesis.

If $c_1 > 1$ then $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|cccc} 0 & & A'x+B' & & \\ \hline 1 & x & 0 & \cdots & 0 \end{array}\right].$$

As above, $\delta_r(A'x+B') = \delta_r$, $\delta_c(A'x+B') = \delta_c$, $j_c(A'x+B') = 0$, $(c_1 > m-1-p$ or $c_1 + 1 > q - 1 - \delta_c)$, and $j_r' = j_r$. Then, $j(A'x + B') \leq j$. Hence,

$$\text{rank } (A'x + B') = \text{rank } (Ax + B) - 1 \geq p - \delta_r + q - 1 - \delta_c - j.$$

By induction hypothesis, the result follows.

Subcase 2.2: Let $j_c > 0$. Then $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|c|c} A'x + B' & 0 & 0 \\ \hline 0 & 0 & A_2x + B_2 \\ \hline 0 & A_1x + B_1 & 0 \end{array}\right],$$

where A_1x+B_1 and A_2x+B_2 have the smallest j_c column and j_r row minimal indices, respectively. Take $\alpha = c_1 + \ldots + c_{j_c}$ and $\beta = r_1 + \ldots + r_{j_r}$. Define $p' = p - (\beta + j_r), q' = q - (\alpha + j_c)$. Observe that $p' \leq m - (\alpha + \beta + j_r)$, $q' \leq n - (\alpha + j_c + \beta)$ and $j_c(A'x + B') = j_r(A'x + B') = 0$. Hence,

$$\text{rank } (A'x+B') = \text{rank } (Ax+B) - \alpha - \beta \geq p - (\beta + j_r) - \delta_r + q - (\alpha + j_c) + \delta_c,$$

and the result follows as a consequence of Subcase 2.1. ■

Let us show the equivalence between the conditions obtained in [9] and condition (11).

Lemma 5.2 *Let $Ax + B$ be an arbitrary pencil. Let $C \in \mathbb{F}^{p\times q}$, rank $C = 0$. Then the following conditions are equivalent*
(a)
a.1) $d_c \leq m - p + n - q + j_c$,
a.2) $d_r \leq m - p + n - q + j_r$.
(b)
b.1) rank $(Ax+B) \geq p - \delta_r + q - \delta_c - j$, where j is defined as in Theorem 3.1.

Proof: Notice that $j = j_c+j_r$ $(0 \leq \text{rank } (Ax+B)-(c_1+\ldots+c_{j_c}+r_1+\ldots+r_{j_r}))$.

(b) → (a): Assume that (b) is satisfied. Condition (b.1) implies that

$$d_c \leq d_c + \text{rank } (Ax + B) - p + \delta_r - q + \delta_c + j \leq$$

$$\leq d_c + d_r - v - p + m - d_r - q + n - d_c + j_c + j_r = m - p + n - q + j_c - (v - j_r),$$

therefore,

$$d_c \leq m - p + n - q + j_c.$$

Condition (a.2) can be obtained analogously.

(a) → (b): Assume that conditions (a) are satisfied. We see next that condition (b.1) is true. We analyze different cases depending on the value of δ_c and δ_r.

Case 1: $\delta_c = q$, $\delta_r = p$. It means that $q \leq n - d_c$ and $p \leq m - d_r$, and the solution follows trivially.

Case 2: $\delta_c = q$, $\delta_r = m - d_r$. Then $d_c \leq n - q$. By definition of j_r,

$$r_1 + \ldots + r_{j_r} \leq n - q \text{ and } r_1 + \ldots + r_{j_r} + j_r \leq p - \delta_r,$$

but

$$r_1 + \ldots + r_{j_r} + r_{j_r+1} > n - q \text{ or } r_1 + \ldots + r_{j_r} + r_{j_r+1} + j_r + 1 > p - \delta_r.$$

The first condition may not happen for $r_1 + \ldots + r_{j_r+1} \leq d_c \leq n - q$. Hence $r_1 + \ldots + r_{j_r} + r_{j_r+1} + j_r + 1 > p - \delta_r$ and

$$v - j_r \leq d_r - (r_1 + \ldots + r_{j_r} + r_{j_r+1} + j_r) \leq m - p.$$

Notice that $j_c = 0$ for $c_1 + \ldots + c_{j_c} + j_c \leq q - \delta_c = 0$ and $j = j_r$. Therefore,

$$p - \delta_r + q - \delta_c - j = p - m + d_r - j_r \leq \text{rank } (Ax + B) = d_r - v,$$

if and only if $v - j_r \leq m - p$, what is true, and condition (b.1) is satisfied.

Case 3: $\delta_c = n - d_c$, $\delta_r = p$. Is dual to case 2.

Case 4: $\delta_c = n - d_c$, $\delta_r = m - d_r$. In this case $m - d_r \leq p$ and $n - d_c \leq q$. Observe that $c_1 + \ldots + c_{j_c+1} > m - p$ or $r_1 + \ldots + r_{j_r+1} > n - q$ may not happen, for if the first condition holds,

$$m - p \geq d_r \geq c_1 + \ldots + c_{j_c} > m - p,$$

what is a contradiction. The behavior is analogous if $r_1 + \ldots + r_{j_r+1} > n - q$.

Assume that $c_1 + \ldots + c_{j_c+1} + j_c + 1 > q - \delta_c = q - n + d_c$ and $r_1 + \ldots + r_{j_r+1} + j_r + 1 > p - \delta_r = p - m + d_r$. Then,

$$p - \delta_r + q - \delta_c - (j_r + j_c) \leq r_1 + \ldots + r_{j_r+1} + c_1 + \ldots + c_{j_c+1} \leq \text{rank } (Ax + B),$$

as desired. ■

6 The case $Ax + B$ regular

In [3] the solution to the problem of the completion of an arbitrary pencil to a regular one is given in terms of an existence condition and for infinite fields. Recently, the authors learnt that the result obtained in [3] has been proven in terms of explicit conditions and for algebraically closed fields [4]. Although the result applies in particular to Problem 1.1 when the pencil $Ax + B$ is regular, we provide here another solution in terms of very simple, explicit conditions, and for algebraically closed fields.

We study first the case when $Ax + B$ is regular having only t infinite elementary divisors, in two steps: when $p = q = \rho$, and without restriction. Then, the general regular case. Notice that if $Ax + B$ is regular, then $d_c = d_r = \text{rank}\ (Ax + B) = m$. It must be remarked that the results of the following Lemmas 6.1 to 6.4 hold for arbitrary fields.

Lemma 6.1 *Let $Ax+B$ be a regular pencil having only t infinite elementary divisors $k_1 \geq \ldots \geq k_t$ as Kronecker invariants. Let $C \in \mathbb{F}^{p\times q}$, rank $C = \rho \geq 1$. Assume that*

$$p = q = \rho, \tag{12}$$

$$p + q \leq m + t. \tag{13}$$

$$p + q \leq m + \rho, \tag{14}$$

$$\rho \leq m - t_1, \tag{15}$$

being t_1 the amount of infinite elementary divisors of degree greater than 1. Then $Ax + B$ is strictly equivalent to

$$\begin{bmatrix} C & * \\ * & * \end{bmatrix}.$$

Proof: Denote by $\alpha_1 \mid \ldots \mid \alpha_\rho$ and $\gamma_1 \mid \ldots \mid \gamma_m$ the homogeneous invariant factors of C and $Ax + B$, respectively. Notice that C is a regular pencil. We have seen above (Remark, page 10) that conditions (14) to (15) amount to the interlacing conditions

$$\gamma_i \mid \alpha_i \mid \gamma_{1+m-p+m-q},\ i = 1, \ldots, \rho.$$

Following [2], we are going to transform the pencil $Ax + B$ into another one $A'x' + B'$ without infinite elementary divisors; then applying Sá-Thompson Theorem (Theorem 2.7) the result follows.

Let us consider the matrix X:

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Define a change of basis (notice that it can be performed in every field)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

With respect to the new basis, $Ax + By = (A+B)x' + By'$. Define

$$p_X(Ax + By) = (A+B)x' + By',$$

$$p_X(Cy) = Cx' + Cy',$$

and for $p(x, y) \in \mathbb{F}[x, y]$,

$$\pi_X(p(x, y)) = p(x', x' + y') = \tilde{p}(x', y').$$

Define $\tilde{\gamma}_i = \pi_X(\gamma_i), \ i = 1, \dots, m$ and $\tilde{\alpha}_i = \pi_X(\alpha_i), \ i = 1, \dots, \rho$. By [2, Lemma 7 to Lemma 10] $\tilde{\gamma}_1 \mid \dots \mid \tilde{\gamma}_m$ are the homogeneous invariant factors of $p_X(Ax + B), \ \tilde{\alpha}_1 \mid \dots \mid \tilde{\alpha}_\rho$ those of $p_X(Cy)$, and

$$\tilde{\gamma}_i \mid \tilde{\alpha}_i \mid \tilde{\gamma}_{1+m-p+m-q}, \ i = 1, \dots, \rho.$$

Moreover, as $\det(A+B) \neq 0$ the pencil $p_X(Ax + B)$ does not have infinite elementary divisors. And as rank $C = \rho = p = q, \ p_X(Cy)$ either not. By Theorem 2.7, the pencil $p_X(Cy)$ can be completed up to $p_X(Ax + By)$. That is, there exist constant matrices Y, Z, W such that

$$\begin{bmatrix} p_X(Cy) & Y \\ Z & W \end{bmatrix} = p_X(Ax + By).$$

By [2, Lemma 6], the transformation p_X is invertible and $(p_X)^{-1} = p_{X^{-1}}$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Performing the inverse transformation we have:

$$p_{X^{-1}}(p_X(Ax + By)) = p_{X^{-1}}(\begin{bmatrix} p_X(Cy) & Y \\ Z & W \end{bmatrix}) = \begin{bmatrix} Cy & p_{X^{-1}}(Y) \\ p_{X^{-1}}(Z) & p_{X^{-1}}(W) \end{bmatrix}.$$

Taking $y = 1$ we obtain the desired result. ■

Lemma 6.2 *Let $Ax+B$ be a regular pencil having only t infinite elementary divisors $k_1 \leq \ldots \leq k_t$ as Kronecker invariants. Let $C \in \mathbb{F}^{p\times q}$, rank $C = \rho \geq 1$. Assume that*

$$p+q \leq m+t. \tag{16}$$

$$p+q \leq m+\rho, \tag{17}$$

$$\rho \leq m-t_1. \tag{18}$$

Then $Ax+B$ is strictly equivalent to

$$\begin{bmatrix} C & * \\ * & * \end{bmatrix}.$$

Proof: We may consider $p < m$, $q < m$ and $\rho < p$ or $\rho < q$. Taking into account Lemma 6.1, we may suppose that $\rho < p$.

If $\rho = 1$, the property follows from the Kronecker canonical form of $Ax+B$.

Assume then that $\rho \geq 2$. Then $p \geq 2$ and $q \geq 2$. We prove by induction on the size of the pencil that the property is true for every m.

It is easy to see that the property is true for $m = 3$. Let $m > 3$.

If $k_1 = 1$, then $Ax+B$ is strictly equivalent to

$$\left[\begin{array}{c|c} 0 & A'x+B' \\ \hline 1 & 0 \end{array}\right].$$

As the following conditions are satisfied:

$$p-1+q-1 \leq m-1+t-1,$$

$$p-1+q-1 \leq m-1+\rho-1,$$

$$\rho-1 \leq m-1-t_1(A'x+B'),$$

by induction hypothesis the result follows.

Assume that $k_1 \geq 2$. Then $Ax+B$ is strictly equivalent to

$$\left[\begin{array}{c|c} 0 & \\ \vdots & A'x+B' \\ 0 & \\ x & \\ \hline 1 & 0 \end{array}\right].$$

We have that

$$p - 1 + q \leq m - 1 + t,$$

$$p - 1 + q \leq m - 1 + \rho.$$

If $\rho \leq m - 1 - t_1(A'x + B')$, according to the induction hypothesis, the result follows. Notice that it is so if $k_1 = 2$.

Assume that $k_1 \geq 3$ and $\rho = m - t_1 = m - t$. Taking into account condition (16) we have that $2(m - t) < p + q \leq m + t$, what implies that $m < 3t$. But $3t \leq m$, what is a contradiction. ■

We will also need the following auxiliary result:

Lemma 6.3 *Let $Ax + B = (A_1x + B_1) \oplus (A_2x + B_2)$ be a pencil, where $A_1x + B_1 \in \mathbb{F}[x]^{m_1 \times n_1}$ has only infinite elementary divisors and $A_2x + B_2 \in \mathbb{F}[x]^{m_2 \times n_2}$ with rank $A_2 = n_2$. Then, for every matrix $Y \in \mathbb{F}^{m_1 \times n_2}$,*

$$\begin{bmatrix} A_1x + B_1 & Y \\ 0 & A_2x + B_2 \end{bmatrix},$$

is strictly equivalent to $Ax + B$.

Proof: We may suppose that $A_1x + B_1$ is in Kronecker canonical form, and $A_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

Assume first that $A_1x + B_1$ has only one infinite elementary divisor

$$A_1x + B_1 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ x & 1 & 0 & \dots & 0 \\ 0 & x & 1 & \dots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & \dots & x & 1 \end{bmatrix} \in \mathbb{F}^{m_1 \times (m_1+1)}.$$

We aim to proof by induction for $j = 1, \dots, m_1$ that $Ax + B$ is strictly equivalent to a pencil of the form

$$\left[\begin{array}{c|c||c} A_1^j x + B_1^j & 0 & 0 \\ \hline Z_j & A_1^{m_1-j} x + B_1^{m_1-j} & Y_j \\ \hline\hline 0 & 0 & A_2x + B_2 \end{array}\right],$$

where $A_1^j x + B_1^j \in \mathbb{F}[x]^{j\times j}$ and $A_1^{m_1-j}x + B_1^{m_1-j} \in \mathbb{F}[x]^{(m_1-j)\times(m_1-j)}$ are pencils in Kronecker canonical form having j and $m_1 - j$ as infinite elementary divisors, respectively, as the only Kronecker invariants, $Z_j \in \mathbb{F}^{(m_1-j)\times j}$ is a matrix of appropriate size of the form

$$Z_j = \begin{bmatrix} 0 & \dots & 0 & x \\ 0 & \dots & 0 & 0 \\ & \ddots & & \\ 0 & \dots & 0 & 0 \end{bmatrix},$$

and $Y_j \in \mathbb{F}^{(m_1-j)\times m_2}$.

If $j = 1$, $A_1^1 x + B_1^1$ has only one infinite elementary divisor $k_1 = 1$: $A_1^1 x + B_1^1 = \begin{bmatrix} 1 \end{bmatrix}$. Through elementary operations on columns, Y can be brought to zero.

Assume that the property is true for a certain j. We see next that it is true for $j + 1$.

Notice that the structure of $A_1^{m_1-j}x + B_1^{m_1-j}$ allows us to say that $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|c|c||c} A_1^j x + B_1^j & 0 & 0 & 0 \\ \hline 0 \ \dots \ 0 \ \ x & 1 & 0 & y^{1j} \\ \hline 0 & \begin{matrix} x \\ 0 \\ \vdots \\ 0 \end{matrix} & A_1^{m_1-j-1}x + B_1^{m_1-j-1} & \begin{matrix} y^{2j} \\ \vdots \\ y^{m_1-jj} \end{matrix} \\ \hline\hline 0 & 0 & 0 & A_2 x + B_2 \end{array}\right],$$

where $A_1^{m_1-j-1}x + B_1^{m_1-j-1} \in \mathbb{F}[x]^{(m_1-j-1)\times(m_1-j-1)}$ has only one infinite elementary divisor and $Y_j = \begin{bmatrix} y^{1j} \\ y^{2j} \\ \vdots \\ y^{m_1-jj} \end{bmatrix}$. By elementary operations on columns, the component 1 in position $(j+1, j+1)$ allows us to make zero the first row of matrix Y_j bringing the components of the second row to polynomials of degree at most one. That is, after the operations on columns Y_j is transformed into

$$\begin{bmatrix} 0 \\ y^{2j} + y^{1j}x \\ y^{3j} \\ \vdots \\ y^{m_1-jj} \end{bmatrix}.$$

Take

$$A_1^{j+1}x + B_1^{j+1} = \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{A_1^j x + B_1^j} & 0 \\ \hline 0 & \ldots & 0 & x & 1 \end{array}\right],$$

and Z_{j+1} equal to a matrix of the form of Z_j, but with one more column and one less row.

As $A_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}$, operating on rows the terms of degree 1 in second row of Y_j can be dropped in every component, resulting a constant matrix

$$\begin{bmatrix} 0 \\ Y_{j+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{y}^{2j} \\ y^{3j} \\ \vdots \\ y^{m_1-jj} \end{bmatrix},$$

and we are done.

Form $j = m_1$ we obtain the desired result.

Assume now that A_1x+B_1 is a pencil in Kronecker canonical form having $k_1 \leq \ldots \leq k_t$ as infinite elementary divisors. Then,

$$A_1x + B_1 = \text{Diag }(K_1, \ldots, K_t).$$

Then we can write

$$\begin{bmatrix} A_1x + B_1 & Y \\ 0 & A_2x + B_2 \end{bmatrix} = \left[\begin{array}{ccc|c} K_1 & & & Y_1 \\ & \ddots & & \vdots \\ & & K_t & Y_t \\ \hline & 0 & & A_2x + B_2 \end{array}\right],$$

for some matrices $Y_1, \ldots Y_t$. Using the first part of this proof, we may reduce successively to zero the blocks $Y_j,\ j = 1, \ldots, t$ obtaining the desired result. ■

The following results are also true.

Corollary 6.4 *Let $Ax + B = (A_1x + B_1) \oplus (A_2x + B_2)$ be a pencil, with $A_2x + B_2 \in \mathbb{F}[x]^{m_2 \times n_2}$, rank $A_2 = n_2$. Assume that one of the following conditions is satisfied*

a) The pencil $A_1x + B_1 \in \mathbb{F}[x]^{m_1 \times n_1}$ has only column minimal indices.

b) The pencil $A_1x + B_1 \in \mathbb{F}[x]^{m_1 \times n_1}$, $n_1 = m_1$, is regular.

Then, for every matrix $Y \in \mathbb{F}^{m_1 \times n_2}$,

$$\begin{bmatrix} A_1x + B_1 & Y \\ 0 & A_2x + B_2 \end{bmatrix},$$

is strictly equivalent to $Ax + B$.

Proof:

a) Following step by step the proof of Lemma 6.3 and changing only the blocks in Kronecker canonical form corresponding to the infinite elementary divisors by blocks corresponding to column minimal indices, the result follows.

b) Combining appropriately the result of Lemma 6.3 and part (a) of this corollary, the result follows. ■

Next theorem gives a solution to Problem 1.1 for regular pencils, when the underlying field is algebraically closed. To prove it we will use Theorem 3.1, Lemmas 6.2 and 6.3 and Theorem 2.10.

Theorem 6.5 *Let $Ax + B$ be a regular pencil in Kronecker canonical form and $Ax + B = (A_1x + B_1) \oplus (A_2x + B_2)$, where all of the elementary divisors of $A_1x + B_1 \in \mathbb{F}[x]^{m_1 \times m_1}$ are infinite and those of $A_2x + B_2 \in \mathbb{F}[x]^{m_2 \times m_2}$ are finite. Let $C \in \mathbb{F}^{p \times q}$, rank $C = \rho$. Then $Ax + B$ is strictly equivalent to a pencil of the form*

$$\begin{bmatrix} C & * \\ * & * \end{bmatrix},$$

if and only if

$$p + q \leq m + t, \tag{19}$$

$$p + q \leq m + \rho, \tag{20}$$

$$\rho \leq m - \max\{i, t_1\}. \tag{21}$$

Proof: The necessity of conditions (19) and (20) is a consequence of conditions (4) and (5) of Theorem 3.1, respectively. Condition (21) comes from the definition of j: Notice that $j = 0$ and $\rho \leq \text{rank }(Ax + B) - \max\{i, t_1\}$. Let us prove the sufficiency.

If $\rho = 0$, the result follows from Theorem 5.1. Assume then that $\rho \geq 1$.

Our aim is to show that for $i = 1, 2$ there exist a matrix $C_i \in \mathbb{F}^{p_i \times q_i}$ of rank ρ_i, $p_1 + p_2 = p$, $q_1 + q_2 = q$, $\rho_1 + \rho_2 \leq \rho$, such that $A_i x + B_i$ is strictly equivalent to a pencil having C_i as a subpencil. Once it is achieved, we will show that the matrix Diag (C_1, C_2) can be completed up to a matrix of rank ρ equivalent to C of the form

$$\begin{bmatrix} C_1 & X \\ 0 & C_2 \end{bmatrix}.$$

Because of Lemma 6.3, the problem will be then solved.

Define (notice that $m_1 + \rho - p_1 - q_1 \geq 0$)

$$p_1 = \min\{p, m_1\},$$

$$q_2 = \min\{q, m_2 - p_2\},$$

$$\rho_2 = \min\{\rho, p_2, q_2, m_2 - i, m_1 + \rho - p_1 - q_1\}$$

$$\rho_1 = \min\{\rho - \rho_2, p_1, q_1, m_1 - t_1\}.$$

Put $p_2 = p - p_1$ and $q_1 = q - q_2$. Then, it is immediate that

$$0 \leq p_1, q_1 \leq m_1, \; 0 \leq p_2, q_2 \leq m_2,$$

$$0 \leq \rho_1 \leq \min\{p_1, q_1\}, \; 0 \leq \rho_2 \leq \min\{p_2, q_2\}.$$

(Observe that $q_1 \leq m_1$ if and only if $q + p \leq m + p_1$, what is immediate if $p_1 = p$ and also true because of (19) if $p_1 = m_1$).

To achieve the goal, according to Lemma 6.2 and Theorem 2.10, it is sufficient to prove that

$$p_1 + q_1 \leq m_1 + t. \tag{22}$$

$$p_1 + q_1 \leq m_1 + \rho_1, \tag{23}$$

$$\rho_1 \leq m_1 - t_1, \tag{24}$$

$$p_2 + q_2 \leq m_2, \tag{25}$$

$$\rho_2 \leq m_2 - i. \tag{26}$$

Notice that under the above definitions, conditions (24) and (26) are satisfied. It is easy to see that $p_1 + q_1 = p_1 + q - q_2 \leq m_1 + t$ is satisfied, what is condition (22).

Concerning condition (23): It is true if $\rho_1 = p_1$, $\rho_1 = q_1$ or $\rho_1 = \rho - \rho_2$. If $\rho_1 = m_1 - t_1$ the property is satisfied as a consequence of condition (22) if we realize that $t \leq m_1 - t_1$.

Taking into account conditions (23) to (24) and Lemma 6.2, there exist a matrix $C_1 \in \mathbb{F}^{p_1 \times q_1}$ of rank ρ_1 such that $A_1x + B_1$ is strictly equivalent to a pencil having C_1 as a submatrix.

Notice that $p_2+q_2 \leq p_2+m_2-p_2$ what is condition (25). Then, conditions (25), (26) and Theorem 2.10 guaranty that there exists a constant submatrix $C_2 \in \mathbb{F}^{p_2 \times q_2}$ of rank ρ_2 which is a subpencil of $A_2x + B_2$.

Let us see that Diag (C_1, C_2) can be completed to a matrix $\begin{bmatrix} C_1 & X \\ 0 & C_2 \end{bmatrix} \in \mathbb{F}^{p \times q}$ of rank ρ, equivalent to C. We analyze different cases according to the values of ρ_1 and ρ_2. It is enough to check that

$$\rho_1 + \rho_2 + \min\{p_1 - \rho_1, q_2 - \rho_2\} = \min\{\rho_2 + p_1, \rho_1 + q_2\} \geq \rho.$$

We have that $\rho_2 + p_1 \geq \rho$ for $\rho_2 = \rho$ and $\rho_2 = p_2$. For the remaining cases of ρ_2, notice that if $p_1 = p$, then $\rho_2 + p \geq \rho$. Hence, assume that $p_1 = m_1$. Then, if $\rho_2 = q_2$, $\rho_2 + p_1 = q_2 + m_1 \geq q \geq \rho$. If $\rho_2 = m_2 - i$, then $\rho_2 + p_1 = m_2 - i + m_1 = m - i \geq m - l \geq \rho$, and if $\rho_2 = m_1 + \rho - p_1 - q_1$, then $\rho_2 + p_1 = m_1 - q_1 + \rho \geq \rho$.

On the other hand, $\rho_1 + q_2 \geq \rho$ if $\rho_1 = \rho - \rho_2$ or $\rho_1 = q_1$. If $\rho_1 = p_1$, we have already seen that $p_1 + q_2 \geq \rho$. If $\rho_1 = m_1 - t_1$, then $\rho_1 + q_2 \geq \rho$ if $q_2 = q$. If $q_2 = m_2 - p_2$, then $\rho_1 + q_2 = m_1 - t_1 + m_2 - p_2 \geq m - l - p_2 \geq \rho$ if $p_2 = 0$ $(p_1 = p)$. If $p_1 = m_1$, then $m_1 - t_1 + m_2 - p + m_1 \geq t + m - p \geq \rho$ from condition (19).

Therefore, $Ax + B$ is strictly equivalent to a pencil having a constant matrix $p \times q$ of rank ρ as a submatrix, as desired. ■

7 The case $Ax + B$ has only nonzero column minimal indices.

In this section we solve the case where the pencil $Ax + B$ has only nonzero column minimal indices. Notice that now $d_r = m = \text{rank }(Ax+B) = \text{rank } A$

and $d_c = m + u$. Observe that the following Lemma 7.1 and Theorem 7.2 hold for arbitrary fields.

Lemma 7.1 *Let $Ax + B \in \mathbb{F}^{m\times n}$ be a pencil in Kronecker canonical form having only nonzero column minimal indices $c_1 \leq \ldots \leq c_u$. Let $C \in \mathbb{F}^{p\times q}$, rank $C = \rho \geq 1$. Assume that*

$$p + q \leq n, \tag{27}$$

$$p + q \leq m + \rho. \tag{28}$$

Then $Ax + B$ is strictly equivalent to a pencil of the form

$$\begin{bmatrix} C & * \\ * & * \end{bmatrix}.$$

Proof: We may suppose that $p < m$, $q < n$. We prove the result by induction on m.

If $m = 2$, then $p = 1$ and the result follows from the Kronecker canonical form. Assume that $m > 2$.

If $c_1 = 1$, then $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|c|c} 0 & 0 & A'x + B' \\ \hline x & 1 & 0 \end{array}\right],$$

where $A'x + B'$ has $u - 1$ column minimal indices. Observe that

$$p - 1 + q - 1 \leq n - 2,$$

$$p - 1 + q - 1 \leq m - 1 + \rho - 1.$$

By induction hypothesis, the result follows.

Let $c_1 \geq 2$. Then $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|cccc} 0 & & A'x + B' & & \\ \hline x & 1 & 0 & \ldots & 0 \end{array}\right],$$

where $A'x + B'$ has u column minimal indices.

If $\rho < p$, we have that

$$p - 1 + q \leq n - 1,$$

$$p - 1 + q \leq m - 1 + \rho,$$

and the result follows by induction hypothesis.

If $\rho = p < q$, then

$$p + q - 1 \leq n - 1,$$

$$p + q - 1 \leq m - 1 + \rho,$$

and the result follows by induction hypothesis.

Let $\rho = p = q$. Then $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|c|cccc} 0 & 0 & & A'x + B' & & \\ \hline x & 1 & & 0 & & \\ \hline 1 & 0 & x & 0 & \dots & 0 \end{array}\right].$$

As

$$p - 1 + q - 1 \leq n - 2,$$

$$p - 1 + q - 1 \leq m - 2 + \rho - 1,$$

the result follows by induction hypothesis. ∎

Now we obtain the desired characterization.

Theorem 7.2 *Let $Ax + B \in \mathbb{F}^{m \times n}$ be a pencil in Kronecker canonical form having only nonzero column minimal indices $c_1 \leq \ldots \leq c_u$. Let $C \in \mathbb{F}^{p \times q}$, rank $C = \rho$. Then $Ax + B$ is strictly equivalent to a pencil of the form*

$$\begin{bmatrix} C & * \\ * & * \end{bmatrix},$$

if and only if

$$p + q \leq n, \tag{29}$$

$$p + q \leq m + \rho + j, \tag{30}$$

where j is defined as in Theorem 3.1.

Proof : The necessity is a consequence of Theorem 3.2. Notice that if the pencil has only column minimal indices, by definition of j_c, $p \leq m - (c_1 + \ldots + c_{j_c})$, then $\rho \leq \text{rank } (Ax + B) - (c_1 + \ldots + c_{j_c})$ and $j = j_c$. In particular, the condition on the rank imposed by the definition of j does not imply additional restrictions.

Let us proof the sufficiency. From condition (29) $q < n$. We may assume that $p < m$. If $\rho = 0$, the result comes from Theorem 5.1. Suppose also that $\rho > 0$.

If $j_c = 0$, the sufficiency follows by Lemma 7.1.

Let $j_c > 0$. The pencil $Ax + B$ is strictly equivalent to

$$\begin{bmatrix} A'x + B' & 0 \\ 0 & A''x + B'' \end{bmatrix},$$

where $A''x + B''$ has $c_1, \ldots, c_{j_c}$ as column minimal indices. Let $\alpha = c_1 + \ldots + c_{j_c}$. By definition of j_c, $p \leq m - \alpha$, $\rho \leq q - (\alpha + j_c)$ and $j_c(A'x + B') = 0$. Notice that

$$p + q - (\alpha + j_c) \leq n - (\alpha + j_c),$$

$$p + q - (\alpha + j_c) \leq m - \alpha + \rho.$$

By Lemma 7.1, $A'x+B'$ is strictly equivalent to a pencil having a $p \times (q - \alpha + j_c)$ matrix C_1 of rank ρ as a submatrix. Then, the pencil $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|cc} * & C_1 & 0 \\ \hline * & * & 0 \\ 0 & 0 & A''x + B'' \end{array}\right],$$

and the result follows. ■

8 The case $Ax + B$ has homogeneous invariant factors and nonzero column minimal indices.

For a pencil $Ax + B \in \mathbb{F}^{m \times n}$ without row minimal indices, $j \leq j_c$. Remind that $\rho \leq \text{rank } (Ax + B) - \max\{i, t_1\} - (c_1 + \ldots + c_j)$. We assume that $d_c = n$ and $d_r = m$, hence $n = m + u$, $m = \text{rank } (Ax + B)$ and $\delta_r = \delta_c = 0$.

Theorem 8.1 *Let $Ax + B \in \mathbb{F}^{m \times n}$ be a pencil in Kronecker canonical form having only homogeneous invariant factors and nonzero column minimal indices $c_1 \leq \ldots \leq c_u$. Let $C \in \mathbb{F}^{p \times q}$, rank $C = \rho$. Then $Ax + B$ is strictly equivalent to a pencil of the form*

$$\begin{bmatrix} C & * \\ * & * \end{bmatrix},$$

if and only if

$$p+q \leq n+t, \tag{31}$$

$$p+q \leq m+\rho+j, \tag{32}$$

where j is defined as in Theorem 3.1.

Proof: The necessity is a consequence of Theorem 3.2. Let us proof the sufficiency. We may assume that $p < m$, $\rho > 0$.

Let $m \geq 2$. The pencil $Ax+B$ is strictly equivalent to a pencil

$$\left[\begin{array}{c|c} A_1x+B_1 & 0 \\ \hline 0 & A_2x+B_2 \end{array}\right],$$

where $A_1x+B_1 \in \mathbb{F}[x]^{m_1\times n_1}$, $n_1 = m_1+u$ contains the column minimal indices of $Ax+B$ and $A_2x+B_2 \in \mathbb{F}[x]^{m_2\times m_2}$ its regular part, $m_1+m_2 = m$, $n_1+m_2 = n$.

(a) Assume first that $j_c = 0$. As in Theorem 6.5, we aim to find $C_i \in \mathbb{F}^{p_i\times q_i}$ of rank ρ_i for $i = 1, 2$, $p_1+p_2 = p$, $q_1+q_2 = q$, $\rho_1+\rho_2 \leq \rho$, such that A_ix+B_i is strictly equivalent to a pencil having C_i as a subpencil. Once it is achieved, we will show that the matrix Diag (C_1, C_2) can be completed up to a matrix of rank ρ equivalent to C of the form

$$\begin{bmatrix} C_1 & X \\ 0 & C_2 \end{bmatrix}.$$

Because of Corollary 6.4, the problem will be then solved.

To achieve the first goal, according to Lemma 7.1 and Theorem 6.5 it is sufficient to prove that

$$p_1+q_1 \leq n_1, \tag{33}$$

$$p_1+q_1 \leq m_1+\rho_1. \tag{34}$$

$$p_2+q_2 \leq m_2+t, \tag{35}$$

$$p_2+q_2 \leq m_2+\rho_2, \tag{36}$$

$$\rho_2 \leq m_2-\max\{i, t_1\}. \tag{37}$$

Put $l = \max\{i, t_1\}$. Define

$$p_1 = \min\{p, m_1, n-q\},\ p_2 = p-p_1.$$

It is clear that $0 \le p_1 \le m_1$ and $p_2 \ge 0$. If $p_1 = p$ or $p_1 = m_1$ then $p_2 \le m_2$; if $p_1 = n-q$, then due to condition (31), $p_2 = p-n+q \le n+t-n = t \le m_2$. Therefore $0 \le p_2 \le m_2$. We analyze different cases according to the value of p_1.

Let $p_1 = p$, then $p_2 = 0$. Let

$$q_2 = \min\{q, m_2\}, q_1 = q - q_2.$$

It is immediate that $0 \le q_2 \le m_2$ and $0 \le q_1 \le n_1$.

If $q_2 = q$, the result follows after Corollary 6.4.

Suppose that $q_2 = m_2$. Let $\rho_1 = \min\{\rho, q_1\}$. Then $\rho_1 \le p_1$ and is well defined.

Observe that $p_1+q_1 = p+q-m_2 \le n_1$ and $p_1+q_1 \le m_1+\rho_1$. By Lemma 7.1, $A_1x + B_1$ is strictly equivalent to a pencil having a $p \times q_1$ submatrix C_1 of rank ρ_1 as a subpencil.

It is easy to see that $\rho_1 + \min\{p - \rho_1, m_2\} \ge \rho$, and the result follows.

Let $p_1 = m_1$. Notice that $m - p = m_2 - p_2$ and $q \le n - m_1$. Define

$$q_2 = \min\{q, m_2, m - p + t\}, \; q_1 = q - q_2.$$

$$\rho_1 = q_1, \; \rho_2 = \min\{\rho - \rho_1, p_2, m_2 - l\}.$$

Observe that $0 \le q_2 \le m_2$ and $q_1 \ge 0$. Notice that $q_1 \le u$. It follows that $0 \le q_1 \le n_1$ and $q_1 \le p_1$. Finally, $q_1 \le \rho$: It is true if $q_2 = q$; if $q_2 = m_2$ then, due to condition (32), $q-m_2 \le m+\rho-p-m_2 = \rho-p_2 \le \rho$. If $q_2 = m-p+t$ then $q - m + p - t \le m + \rho - m - t \le \rho$. Therefore, they are well defined.

We show next that they satisfy conditions (33) to (37).

It is so for conditions (34) and (37). Observe that $p_1+q_1 \le m_1+u = n_1$, what is condition (33). We have that $p_2+q_2 \le p_2+m_2-p_2+t$ and condition (35) is also satisfied.

Let us see that $p_2 + q_2 \le m_2 + \rho_2$: It is true if $\rho_2 = p_2$. If $\rho_2 = \rho - \rho_1$, then $p_2 + q_2 \le m_2 + \rho - q_1$ if and only if $p + q \le m + \rho$, what is condition (32).

It is immediate to see that $\rho_1 + \rho_2 + \min\{p_1 - \rho_1, q_2 - \rho_2\} = \min\{p_1 + \rho_2, q_2 + \rho_1\} \ge \rho$. And the result follows.

Let $p_1 = n - q$. We assume that $n - q < m_1$. Then $n - m_1 = m_2 + u < q$. The following values are well defined.

$$q_2 = m_2, q_1 = q - q_2.$$

$$\rho_2 = p_2, \ \rho_1 = \min\{\rho - \rho_2, q_1\}.$$

Notice that $\rho_2 = p_2 = p - n + q \leq \rho$.

It is easy to see that $p_1 + q_1 = n_1$, $p_2 + q_2 = m_2 + \rho_2$ and that conditions (34), (35) and (37) are satisfied.

It only remains to be proven that $\rho_1 + \rho_2 + \min\{p_1 - \rho_1, q_2 - \rho_2\} = \min\{\rho_2 + p_1, \rho_1 + q_2\} \geq \rho$. But $\rho_2 + p_1 = p_2 + p_1 \geq \rho$. Finally, $\rho_1 + q_2 \geq \rho$ if $\rho_1 = q_1$ and if $\rho_1 = \rho - \rho_2$, $\rho_1 + q_2 = \rho - \rho_2 + q_2 \geq \rho$. The result is then satisfied.

(b) Assume that $j_c > 0$. The pencil $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|c} A'x + B' & 0 \\ \hline 0 & A''x + B'' \end{array}\right],$$

where $A''x+B''$ has $c_1, \ldots, c_j$ as column minimal indices. Let $\alpha = c_1 + \ldots + c_j$. By definition of j_c and j, $p \leq m - \alpha$, $\rho \leq q - (\alpha + j_c)$, $j_c' = j_c(A_1x + B_1) = 0$, $j' = j(A_1x + B_1) = 0$ and $\rho \leq m - \alpha - \max\{i, t_1\}$. Notice that

$$p + q - (\alpha + j) \leq m - (\alpha + j) + u,$$

$$p + q - (\alpha + j) \leq m - \alpha + \rho.$$

By the result of part (a) $A'x + B'$ is strictly equivalent to

$$\begin{bmatrix} * & C_1 \\ * & * \end{bmatrix},$$

where $C_1 \in \mathbb{F}^{p \times (q - \alpha + j)}$, rank $C_1 = \rho$. Then, $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|c|c} * & C_1 & 0 \\ \hline * & * & 0 \\ \hline 0 & 0 & A''x + B'' \end{array}\right].$$

Notice that the top right submatrix $\begin{bmatrix} C_1 & 0 \end{bmatrix}$ is the desired one. The result follows. ■

9 The general case.

Let $\mathbb{F}$ be an algebraically closed field. In this section we prove the result when $Ax+B$ is an arbitrary pencil. We study first some particular cases.

Lemma 9.1 *Let $Ax+B \in \mathbb{F}^{m\times n}$. Let $C \in \mathbb{F}^{p\times q}$, rank $C = \rho$. Assume that $j_c = j_r = 0$, $d_c = n$, $d_r = m$. Assume that $\rho = p = q$. If*

$$2\rho \leq m + n - \textit{rank } A, \tag{38}$$

$$\rho \leq \textit{rank } (Ax+B) - \max\{i, t_1\}, \tag{39}$$

then $Ax+B$ is strictly equivalent to a pencil of the form

$$\begin{bmatrix} C & * \\ * & * \end{bmatrix}.$$

Proof: The pencil $Ax+B$ can be written as $(A_1x+B_1)\oplus(A_2x+B_2)$, where $m_1+m_2 = m$, $n_1+n_2 = n$, $A_1x+B_1 \in \mathbb{F}[x]^{m_1\times n_1}$ has only column minimal indices and $A_2x+B_2 \in \mathbb{F}[x]^{m_2\times n_2}$ has homogeneous invariant factors and row minimal indices. We aim to proceed as in Theorem 8.1. In this case it is enough to find ρ_1, ρ_2 such that taking $p_1 = q_1 = \rho_1, p_2 = q_2 = \rho_2$, the conditions of Lemma 7.1 and Theorem 8.1 are satisfied.

Define

$$\rho_1 = \min\{\rho, [\frac{n_1}{2}], m_1\},$$

$$\rho_2 = \min\{\rho - \rho_1, [\frac{m_2+t}{2}], n_2 - l\},$$

with $l = \max\{i, t_1\}$. They are well defined and satisfy the desired conditions

$$2\rho_1 \leq n_1, \tag{40}$$

$$2\rho_2 \leq m_2 + t. \tag{41}$$

Notice that as

$$\rho_2 \leq \text{rank } (A_2x+B_2) - \max\{i, t_1\}, \tag{42}$$

$j' = j(A_2x+B_2 = 0$. It only remains to prove that

$$\rho_1 + \rho_2 + \min\{m_1 - \rho_1, n_2 - \rho_2\} \geq \rho.$$

Let us see that $m_1 + \rho_2 \geq \rho$:
It is so if $\rho_2 = \rho - \rho_1$, for $m_1 - \rho_1 + \rho \geq \rho$.

If $\rho_2 = [\frac{m_2+t}{2}] = \frac{m_2+t}{2}$, then $m_1 + \frac{m_2+t}{2} \geq \rho$ if and only if $2m_1 + m_2 + t \geq 2\rho$, what is true because of condition (38), for $2m_1 + m_2 + t \geq m + t + u \geq 2\rho$.
If $\rho_2 = \frac{m_2+t-1}{2}$, then $m_1 + \frac{m_2+t-1}{2} \geq \rho$ if and only if $m + t + m_1 - 1 \geq 2\rho$, what from condition (38) is true but in the case that $m_1 + u + m_2 + t = 2\rho$. As $m_2 + t$ is odd, $m_1 + u$ must be odd. Then $\frac{m_1+u+1}{2} + \frac{m_2+t-1}{2} = \rho$. Observe that $\frac{m_1+u+1}{2} \leq m_1$ for $u < m_1$, hence $m_1 + \frac{m_2+t-1}{2} \geq \frac{m_1+u+1}{2} + \frac{m_2+t-1}{2} = \rho$ and the property is true.
If $\rho_2 = n_2 - l$, then $m_1 + n_2 - l \geq \text{rank }(Ax+B) - l \geq \rho$ from condition (39).

On the other hand, $\rho_1 + n_2 \geq \rho$: It is so if $\rho_1 = \rho$.
If $\rho_1 = [\frac{n_1}{2}] = \frac{n_1}{2}$ then $\frac{n_1}{2} + n_2 \geq \rho$ if and only if $n_1 + 2n_2 \geq 2\rho$, what is true from condition (38), for $n_1 + 2n_2 = n + n_2 \geq n + t + v \geq 2\rho$.
If $\rho_1 = \frac{n_1-1}{2}$ then it must happen that $n_1 - 1 + 2n_2 = n + n_2 - 1 \geq 2\rho$. If it is not the case, $n_1 - 1 + 2n_2 < 2\rho$. Then $t + v - 1 + n < 2\rho \leq t + v + n$, what implies that $2\rho = n_1 + n_2 + t + v$, from where $n_2 + t + v$ must be odd and $n_2 - 1 \geq v + t$. Therefore, $n_1 - 1 + 2n_2 = n + n_2 - 1 \geq n + t + v \geq 2\rho$ as desired.
If $\rho_1 = m_1$, then $m_1 + n_2 \geq n - u - \max\{i, t_1\} \geq \rho$. ■

Lemma 9.2 *Let $Ax + B \in \mathbb{F}^{m\times n}$. Assume that $d_c = n$, $d_r = m$. Let $C \in \mathbb{F}^{p\times q}$, rank $C = \rho$. If*

$$\text{rank } A \leq m - p + n - q, \tag{43}$$

$$\text{rank } (Ax + B) \geq p + q - \rho. \tag{44}$$

$$\rho \leq \text{rank } (Ax + B) - \max\{i, t_1\}, \tag{45}$$

then $Ax + B$ is strictly equivalent to a pencil of the form

$$\begin{bmatrix} C & * \\ * & * \end{bmatrix}.$$

Proof: We may suppose that $\rho < q$, $p < m, q < n$, $u > 0$ (if $u = 0$, the result comes after Theorem 8.1) and $\rho > 0$.

We are going to prove its sufficiency by induction, analyzing different cases:

<u>Case 1</u>: $c_1 = 1$. In this case $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|c|c} 0 & 0 & A'x + B' \\ \hline x & 1 & 0 \end{array}\right].$$

The following conditions are satisfied:

$$\text{rank } A - 1 \leq m - 1 - (p-1) + n - 2 - (q-1),$$

$$\rho - 1 \leq \text{rank } (Ax+B) - 1 - \max\{i, t_1\} = \text{rank } (A'x+B') - \max\{i, t_1\},$$

$$\text{rank } (Ax+B) - 1 \geq p - 1 + q - 1 - (\rho - 1).$$

By induction hypothesis, the result follows.

Case 2: $c_1 > 1$. Then $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|ccc} 0 & & A'x+B' & \\ \hline 1 & x & 0 & 0 \end{array}\right].$$

The following conditions are satisfied:

$$\text{rank } A - 1 \leq m - 1 - p + n - 1 - (q-1),$$

$$\text{rank } (A'x+B') = \text{rank } (Ax+B) - 1 \geq p + q - 1 - \rho.$$

If $\rho \leq \text{rank } (Ax+B) - 1 - \max\{i, t_1\} = \text{rank } (A'x+B') - \max\{i, t_1\}$ fulfills, the result follows by induction hypothesis.

Otherwise, $\rho = \text{rank } (Ax+B) - \max\{i, t_1\}$. Denote by $w = \text{rank } (Ax+B)$ and $l = \max\{i, t_1\}$. Let us analyze this case. From condition (43) we obtain

$$2\rho = 2(w-l) \leq p + q \leq m + n - \text{rank } A = w + v + w + u - (w - t),$$

hence

$$w \leq 2l + u + v + t. \tag{46}$$

We then have that

$$2u + v + t + t_1 + i \leq w \leq 2l + u + v + t$$

that is $u + t_1 + i \leq 2l$. As a consequence, if $i = t_1$ the former inequality turns into $u \leq 0$, what is a contradiction. Therefore, $i = t_1$ may not occur if $w = \rho - l$.

Assume that $i \neq t_1$. Let us see what happens to the equality $\rho = w - l$ when detached one row and one column from the pencil. We analyze the possible different types of blocks in the Kronecker canonical form of $Ax + B$ that can be involved.

a) Assume that there exists an infinite elementary divisor of degree one. Then

$$Ax + B \overset{st}{\sim} \left[\begin{array}{c|c} 0 & A'x + B' \\ \hline 1 & 0 \end{array}\right].$$

The following conditions are satisfied

$$\text{rank } A - 1 \leq m - 1 - (p - 1) + n - 1 - (q - 1),$$

$$\rho - 1 \leq \text{rank } (Ax + B) - 1 - \max\{i, t_1\},$$

$$\text{rank } (Ax + B) - 1 \geq p - 1 + q - 1 - (\rho - 1),$$

and the result follows by induction hypothesis.

b) Assume that there exists an infinite elementary divisor of degree two. Then

$$Ax + B \overset{st}{\sim} \left[\begin{array}{c|cc} 0 & 0 & A'x + B' \\ 0 & 1 & 0 \\ \hline 1 & x & 0 \end{array}\right].$$

If $l = t_1 > i$, then $\rho \leq w - 1 - (l - 1)$ and the result follows by induction hypothesis.
If $l = i > t_1$, then $Ax + B$ has finite invariant factors, situation which analyzed below.

c) Assume that $Ax + B$ has a finite invariant factor of degree 1, $x - a$. Then

$$Ax + B \overset{st}{\sim} \left[\begin{array}{c|c} 0 & A'x + B' \\ \hline x - a & 0 \end{array}\right].$$

Then if $l = i > t_1$ we have that $\rho \leq w - 1 - (l - 1)$ and the induction applies.
If $l = t_1 > i$, then $t_1 > 0$. The result follows from (a).

d) Assume that all of the nontrivial finite invariant factors have degrees greater than 1 and all of the infinite elementary divisors have degrees greater than 2 ($k_1 \geq 3$). Then from (46) we have that

$$2u + v + t + 2t_1 + 2i \leq w \leq 2l + u + v + t.$$

If $l = i > t_1$, then $u + 2t_1 \leq 0$, which is impossible. If $l = t_1 > i$, then $u + 2i \leq 0$, which is also impossible.

e) Assume that there are no homogeneous invariant factors. Then from (46) we have that

$$2u + v \le u + v,$$

what may not happen. ■

Next theorem shows the result to Problem 1.1 for the general case.

Theorem 9.3 *Let $Ax + B \in \mathbb{F}^{m\times n}$. Let $C \in \mathbb{F}^{p\times q}$, rank $C = \rho$. Then there exists a pencil strictly equivalent to $Ax + B$ containing C as a subpencil if and only if*

$$\text{rank } A \le d_r - (p - \delta_r) + d_c - (q - \delta_c), \tag{47}$$

$$\text{rank } (Ax + B) \ge p - \delta_r + q - \delta_c - \rho - j, \tag{48}$$

where j is defined as in Theorem 3.1.

Proof: The necessity comes from Theorem 3.2. We proof the sufficiency.

Remind that $\delta_c = \min\{q - \rho, n - d_c\}$, $\delta_r = \min\{p - \rho, m - d_r\}$. By definition of j,

$$\rho \le \text{rank } (Ax + B) - \max\{i, t_1\} - (d_1 + \ldots + d_j),$$

the integers $\{d_1, \ldots, d_{j_c+j_r}\}$ being an increasing reordering of $\{c_1, \ldots, c_{j_c}, r_1, \ldots, r_{j_r}\}$. Let $\tilde{j}_c$, $\tilde{j}_r$ be such that $d_1, \ldots, d_j$ are the values of $c_1, \ldots, c_{\tilde{j}_c}, r_1, \ldots, r_{\tilde{j}_r}$. Take $\tilde{\alpha} = c_1 + \ldots + c_{\tilde{j}_c}$ and $\tilde{\beta} = r_1 + \ldots + r_{\tilde{j}_r}$. Then $Ax + B$ is strictly equivalent to

$$\left[\begin{array}{c|c|c|c} A'x + B' & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \bar{A}''x + \bar{B}'' \\ \hline 0 & 0 & A''x + B'' & 0 \end{array}\right],$$

where $A''x + B'' \in \mathbb{F}^{\tilde{\alpha}\times(\tilde{\alpha}+\tilde{j}_c)}$ and $\bar{A}''x + \bar{B}'' \in \mathbb{F}^{(\tilde{\beta}+\tilde{j}_r)\times\tilde{\beta}}$ have $c_1, \ldots, c_{\tilde{j}_c}$ and $r_1, \ldots, r_{\tilde{j}_r}$ as column and row minimal indices respectively, the second block column corresponds to the $n - d_c$ zero columns of the whole pencil, the second block row to the $m - d_r$ zero rows, and $A'x + B' \in \mathbb{F}^{m'\times n'}$ with $m' = d_r - (\tilde{\alpha} + \tilde{\beta} + \tilde{j}_r)$, $n' = d_c - (\tilde{\alpha} + \tilde{j}_c + \tilde{\beta})$. Hence, $d_c' = d_c(A'x + B') = d_c - (\tilde{\alpha} + \tilde{\beta} + \tilde{j}_c) = n'$ and $d_r' = d_r(A'x + B') = d_r - (\tilde{\alpha} + \tilde{\beta} + \tilde{j}_r) = m'$.

Put $p' = p - (\tilde{\beta} + \tilde{j}_r + \delta_r)$, $q' = q - (\tilde{\alpha} + \tilde{j}_c + \delta_c)$. By definition of $\tilde{j}_c$ and $\tilde{j}_r$ it is easy to see that $\rho \le p' \le m'$ and $\rho \le q' \le n'$. From condition (47) we have that

$$\text{rank } A' \le d_r' - p' + d_c' - q. \tag{49}$$

Let us calculate $j' = j(A'x + B')$. As

$$m' - p' \le m - p - \tilde{\alpha},$$

$$q' - \rho = q - (\tilde{\alpha} + \tilde{j}_c + \delta_c) - \rho.$$

if $j'_c = j_c(A'x + B')$ then $0 \le j'_c \le j_c$, and the column minimal indices it involves are among $c_{\tilde{j}_c+1}, \ldots, c_{j_c}$. In the same way, $0 \le j'_r = j_r(A'x+B') \le j_r$, and the choice of the row minimal indices is analogous. Let $\bar{d}_1, \ldots, \bar{d}_{j'_c+j'_r}$ an increasing reordering of $c_{\tilde{j}_c+1}, \ldots, c_{\tilde{j}_c+j'_c}, r_{\tilde{j}_r+1}, \ldots, r_{\tilde{j}_r+j'_r}$. Noticing that rank $(A'x + B') =$ rank $(Ax + B) - (\tilde{\alpha} + \tilde{\beta})$, it results that

$$j' = \max\{k \in \{0, 1, \ldots, j'_c+j'_r\} : \rho \le \text{rank } (A'x+B') - l - (\bar{d}_1 + \ldots + \bar{d}_k)\} = 0.$$

From condition (48), we obtain

$$\text{rank } (A'x + B') \ge p' + q' - \rho. \tag{50}$$

Taking into account conditions (49) and (50) and applying the former Theorem, there exists a submatrix C_1, rank $C_1 = \rho$, such that $A'x + B'$ is strictly equivalent to a pencil containing C_1 as a subpencil. Then, $Ax + B$ is strictly equivalent to a matrix of the form

$$\left[\begin{array}{cc|c|c} C_1 & * & 0 & 0 \\ * & * & 0 & 0 \\ \hline 0 & 0 & 0 & \bar{A}''x + \bar{B}'' \\ \hline 0 & 0 & A''x + B'' & 0 \end{array}\right].$$

Observe that the whole pencil contains a $p \times q$ matrix C of rank ρ, as desired.
$\blacksquare$

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