Reduction of an Ito´s diffusion input-output model for the determination of the mean-square stability*

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Abstract

Through the use of detailed algorithms we propose a dimension reduction of a multisectoral Ito's difussion linear input-output model. In particular, when we consider this dynamical economic system, a system of differential equations with symmetric state variables for the investigation of the mean-square stability is associated. From this system a $d^2 \times d^2$ matrix is obtained. In this work we proposed a general algorithm that transforms the $d^2 \times d^2$ matrix to one of order $\frac{d(d+1)}{2} \times \frac{d(d+1)}{2}$, conserving the same eigenvalues information. This reduction algorithm allows us to compute the eigenvalues for large scale input-output systems.

Keywords. Dynamical systems, $\lambda-$matrices, algorithm, rational points on curves.

1. Introduction

Consider the multidimensional linear Ito´s processes expressed in the notation of differentials

$$\begin{aligned} dY_t &= (A(t)Y_t + a(t))\,dt + \sum_{i=1}^{d} (B_i(t)Y_t + b_i(t))\,dW_t^i, \\ Y_{t_0} &= Ec; \end{aligned}$$

where $A(t)$ and $B_i(t)$ are real matrix functions of $d \times d$, $a(t)$ and $b_i(t)$, Y_t are real functions of $d \times 1$. W_t^i are independent sources of noise with *normal* $(0,t)$ distribution and $W_t = \left(W_t^1, ..., W_t^m\right)$ is the Wiener's stochastic process of dimension m with *normal* $\left(\bar{0}, tId\right)$ distribution where d is a positive integer.

The Ito's theorem can be applied to the vector Y_t , being obtained from the deterministic dynamical system, to investigate the dynamics of the process in the mean

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$$
\begin{aligned}
\dot{m}_t &= A(t)m_t + a(t), \\
m_{t_0} &= Ec.
\end{aligned}
$$

If the matrix of moments $\rho = (\rho_{ij}) = E\left(Y_t^i Y_t^j\right)$, with Y_t^k components of Y_t is defined and the multidimensional Ito's formula is applied to the matrix YY', we will get the matrix of the deterministic dynamical system for the mean-square stability

$$
\begin{aligned}
\frac{d\rho}{dt} &= A(t)\rho + \rho A(t)' + a(t)m' + ma(t)' + \sum_{i=1}^{d} B_i(t)\rho B_i(t) \qquad (1) \\
&+ B_i(t)mb_i(t)' + b_i(t)m'B_i(t)' + b_i(t)b_i(t)', \\
\rho(t_0) &= Ecc'.
\end{aligned}
$$

For more details of this Ito's diffusion process the reader can see [2].

In this paper some detailed algorithms are given for the reduction of an economic model of the type of equation 1.

For different types of systems, there are many different results about the mean-square stability, for example [12] proposes a method to deal with the mean-square exponential stability of impulsive stochastic difference equations. [6] investigates the mean-square stability for numerical integration of Milstein-type methods. [3] consider the mean-square stability for networked control systems and [11] and [1] consider the same concept for another numerical methods. [9] gives detailed algorithms to reduce a linear dynamical model.

The paper is organized as follows. In section 2 the dynamic multisectorial stochastic models are given. In section 3 we identify the systems in the mean and the mean-square under the Ito's notation. Using Jordan's matrices and, under the hypothesis of asymptotic stability in the mean, the dynamical system in the mean-square is simplified.The methods employed to find the formulas are a beautiful combination of number theory and numerical experimentation. In Section 4 the algorithm is given. Section 5 and 6 contains auxiliary lemmas.

2. The model

We choose the following notation in the description of the model. Let d be an even positive whole number. The endogenous variables are contained in the vectors P, X of prices and quantities of $\frac{d}{2} \times 1$. d_{11}, d_{22}, d_{12}, d_{21} are the diagonal matrices of $\frac{d}{2} \times \frac{d}{2}$, positive or zero, of adjustment speeds, $A(\tau)$ is the matrix of $\frac{d}{2} \times \frac{d}{2}$ of technical coefficients that determines the intermediate demand for the years τ. These matrices can be interpreted as matrices of parameters that vary for each year. g is the growth rate for the consumption and r is the profit rate of the companies, C is the vector of $\frac{d}{2} \times 1$ of final consumption and W is the vector of $\frac{d}{2} \times 1$ of wage payments by production unit.

$\xi_W(t),\ \xi_C(t) \in R^{\frac{d}{2}}$ are white noise vectors

$$\xi_k(t)' = \left(\xi_{1k}(t), \xi_{2k}(t), ..., \xi_{\frac{d}{2}k}(t)\right)',\ k = C, W$$

For each $i = 1, 2, ...\frac{d}{2}$, and for a time $t \in R$, we will suppose that $\xi_i(t)$,are independent stochastic variables that are normally distributed with mean $E(\xi_i(t)) = 0$, and covariance matrix given by:

$$\begin{aligned} Cov_\xi(t_1, t_2) &= E\left(\xi_i(t_1)\xi_j(t_2)\right), \\ &= \left\{ \begin{array}{c} \delta(t_2 - t_1) \text{ if } i = j \\ 0, \text{ other case} \end{array} \right. ; \end{aligned}$$

where δ, it is the Dirac's delta function of impulse. The δ function depends on the difference $t_2 - t_1$, the perturbation is stationary in the wide sense.

$P * \xi_W(t)$ and $X * \xi_C(t)$, are the stochastic perturbations that act according to the product $*$:

$$\begin{aligned} (P * \xi_W(t))' &= \left(P_1\xi_{1W}(t), P_2\xi_{2W}(t), ..., P_{\frac{d}{2}}\xi_{\frac{d}{2}W}(t)\right)', \\ (X * \xi_C(t))' &= \left(X_1\xi_{1C}(t), X_2\xi_{2C}(t), ..., X_{\frac{d}{2}}\xi_{\frac{d}{2}C}(t)\right)'. \end{aligned}$$

These vectors can be interpreted as the risks in prices and quantities.

The apostrophe $'$ is referred to the transposed matrix.

Defining the corresponding adjustment matrixes d_{ij}, the different economic schemes under consideration (next paragraphs) are contained in the following set of stochastic differential equations:

$$\begin{aligned} \frac{dX}{dt} &= d_{11}\left(A(\tau)X + gA(\tau)X + C - X\right) - \\ &\quad d_{12}\left(A(\tau)'P + rA(\tau)'P + W - P\right) + X * \xi_C(t), \\ \frac{dP}{dt} &= d_{21}\left(A(\tau)X + gA(\tau)X + C - X\right) + \\ &\quad d_{22}\left(A(\tau)'P + rA(\tau)'P + W - P\right) + P * \xi_W(t) \end{aligned} \tag{2}$$

This stochastic dynamical system can be written by defining a real function $\zeta(t)$ of $d \times 1$, $\zeta_t = \left(\zeta_t^1, ..., \zeta_t^d\right)$, and then $\zeta(t)' = (X(t), P(t))'$, and $\xi(t) = (\xi_C(t), \xi_W(t))$, $*$ is the product that was previously defined, $\mathbf{A}$ is a matrix of $d \times d$, $\mathbf{B}$ is a vector of $d \times 1$, $\xi(t)$ is a real function of $d \times 1$. Therefore

$$\begin{aligned} \frac{d\zeta(t)}{dt} &= \mathbf{A}\zeta(t) + \mathbf{B} + \zeta(t) * \xi(t), \\ \zeta(t_0) &= c. \end{aligned} \tag{3}$$

where

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} d_{11} & -d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} (1+g)\,A - I & 0 \\ 0 & (1+r)\,A' - I \end{pmatrix}, \\
\mathbf{B} &= \begin{pmatrix} d_{11} & -d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} C \\ W \end{pmatrix}.
\end{aligned}
$$

With the previous notation, the following models can be defined.

The first one, the Neoclassical model, arises when we define diagonal matrices like $d_{11} = d_{22} = 0$; and matrices $d_{12} \neq 0, d_{21} \neq 0$. This model considers the following principles in a free market economy:

1) The law of the excessive rent: the production increases $\left(\frac{dX}{dt} > 0\right)$ when the price $(d_{12}P)$ is greater than the cost $d_{12}\left(A(\tau)' P + rA(\tau)' P + W\right) + X * \xi_C(t)$, and vice versa.

2) The law of the excessive demand: the price increases $\left(\frac{dP}{dt} > 0\right)$ if the demand $d_{21}\left(A(\tau)X + gA(\tau)X + C\right) + P * \xi_W(t)$ is superior to the supply $(d_{21}X)$ and vice versa.

3) Stochastic perturbations $P * \xi_W(t)$, $X * \xi_C(t)$ in the vectors of prices and quantities that reflect unpredictable demands and costs or uncertainties, derivatives of stochastic external effects in the economical system.

In the Keynesian model we have $d_{12} = d_{21} = 0$; and $d_{11} \neq 0, d_{22} \neq 0$

When we consider the production dependent on the effective demand and the prices dependent on the costs and the work, according to Keynes, it is necessary an adjustment of the prices and the quantities. This mechanism, takes into account the reactions from the global system, because the production depends on the demand and it depends on the occupation; the production costs affect the prices and they do not depend on the balance between supply and demand. In addition, the same stochastic perturbations in wages and consumptions are considered as before, derived from the Keynesian theory and they can be interpreted as additional effects in the demand and the costs.

In fact, the following Keynesian economic principles are being considered:

1) The production increases $\left(\frac{dX}{dt} > 0\right)$ if the demand $d_{11}\left(A(\tau)X + gA(\tau)X + C\right) + X * \xi_W(t)$ is superior to the supply $(d_{11}X)$ and vice versa.

2) The price increases if the costs $d_{22}\left(A(\tau)' P + rA(\tau)' P + W\right) + P * \xi_C(t)$, are superior to the price $(d_{22}P)$ and vice versa.

3) The stochastic effects, $X * \xi_W(t)$, $P * \xi_C(t)$, are demands that depend on the supply and the costs depend on the price.

We will refer as the Post-Keynesian model where all the matrices of adjustment are non-zero. This model is a combination of the Neoclassical and the Keynesian ideas and we apply both rent and demand excess laws.

i) The price increases $\left(\frac{dP}{dt} > 0\right)$ if the demand $\left(d_{11}\left(A(\tau)X + gA(\tau)X + C\right)\right)$ added to the cost $\left(d_{12}\left(A(\tau)' P + rA(\tau)' P + W\right)\right)$ and the additional cost $(P * \xi_W(t))$ is superior to the price $(d_{12}P)$ added to the supply $(d_{11}X)$,

ii) The production increases $\left(\frac{dX}{dt} > 0\right)$if the price $(d_{12}P)$ added to the demand $(d_{11}A(\tau)X + gA(\tau)X + C)$ and the additional demand $(X * \xi_C(t))$ is superior to the supply $(d_{11}X)$ added to the costs $\left(d_{12}\left(A(\tau)' P + rA(\tau)' P + W - P\right)\right)$.

3. The system in the mean and the mean-square

In the case of interest $a(t) = \mathbf{B}$, $A(t) = \mathbf{A}$, $B_i = E_{ii}$, $b_i = \bar{0}$. We also assume that c is a constant vector. In the case of interest $m = d$.

E_{ii} is a matrix of $d \times d$ with all elements zero in the position (i, i), where the matrix has a 1.

According to Ito's interpretation, the set of differential equations 3 can be written in the form of a stochastic integral of the diffusion process ζ_t

$$\zeta(t) = c + \int_{t_0}^{t} (\mathbf{A}\zeta(s) + \mathbf{B})ds + \int_{t_0}^{t} \overset{\wedge}{\zeta}_t dW_s,$$

equivalently, it can be rewritten in differential notation as:

$$\begin{aligned} d\zeta_t &= (\mathbf{A}\zeta_t + \mathbf{B})\, dt + \overset{\wedge}{\zeta}_t dW_t \\ &= (\mathbf{A}\zeta_t + \mathbf{B})\, dt + \sum_{i=1}^{d} E_{ii}\zeta_t dW_t^i \end{aligned} \tag{4}$$

with the drift vector

$$\mathbf{A}\zeta_t + \mathbf{B}$$

and the diffusion matrix of $d \times d$

$$\left(\overset{\wedge}{\zeta}_t\right)^2 = \left(diag\left(\zeta_t^1, ..., \zeta_t^d\right)\right)^2$$

The dynamical system for the mean can be solved and we obtain,

$$m_t = e^{\mathbf{A}(t-t_0)}c + e^{\mathbf{A}(t-t_0)} \int_{t_0}^{t} e^{-\mathbf{A}(s-t_0)}\mathbf{B}\; ds$$

$e^{\mathbf{A}t}$ is the exponential of the matrix $\mathbf{A}t$.

By simplicity we have taken $t_0 = 0$. Replacing the solution of the dynamical system in the mean on the dynamical system of the mean-square, we get:

$$\begin{aligned} \frac{d\rho}{dt} &= \mathbf{A}\rho + \rho\mathbf{A}' + \mathbf{B}\left(e^{\mathbf{A}t}c + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}s}\mathbf{B}\; ds\right)' + \left(e^{\mathbf{A}t}c + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}s}\mathbf{B}\; ds\right)\mathbf{B}' \\ &\quad + \sum_{i=1}^{n} E_{ii}\rho E_{ii}, \end{aligned}$$

There are two important cases is this system; when the matrix $\mathbf{A}$ is invertible and when it is not invertible.

When the matrix $\mathbf{A}$ is invertible, the previous formula can be reduced to:

$$
\begin{aligned}
\frac{d\rho}{dt} &= \mathbf{A}\rho + \rho\mathbf{A}' + \mathbf{B}\left(e^{\mathbf{A}t}c - \mathbf{A}^{-1}\mathbf{B}\right)' + \left(e^{\mathbf{A}t}c - \mathbf{A}^{-1}\mathbf{B}\right)\mathbf{B}' + \\
& diag\left(\rho_{11}, \rho_{22}, ..., \rho_{nn}\right)
\end{aligned}
$$

As the matrix $\mathbf{A}$ can be non-invertible, it is preferable to have the more general simplification of the previous set of differential equations when the Jordan's canonical form of the matrix $\mathbf{A}$ is used**.** This simplification is equivalent to the case when the matrix is invertible, it is, when the eigenvalues of the matrix $\mathbf{A}$ have a negative real part (in numerical simulations with input-output matrices for the mexican economy we have both Keynesian and post-keynesian economies).

If α_i are the eigenvalues of the matrix $\mathbf{A}$,thus the i-th Jordan's block will have the form:

$$
J_i = \begin{pmatrix}
\alpha_i & & & & \\
1 & \alpha_i & & & \\
 & 1 & \alpha_i & & \\
 & & \ddots & \ddots & \\
 & & & 1 & \alpha_i
\end{pmatrix}
$$

and the matrix $\mathbf{A}$ can be written in terms of the change matrix of base P, by Jordan in the usual form $\mathbf{A} = PJP^{-1}$, where $J = diag\left(J_1, J_2, \cdots, J_l\right)$, then $e^{At} = Pe^{Jt}P^{-1}$, where $e^{Jt} = diag\left(e^{J_1 t}, e^{J_2 t}, \cdots, e^{J_l t}\right)$ and, in addition:

$$
e^{J_i t} = e^{\alpha_i t} \begin{pmatrix}
1 & & & & \\
\frac{1}{1!} & 1 & & & \\
\frac{1}{2!} & \frac{1}{1!} & 1 & & \\
\vdots & \vdots & & \ddots & \\
\frac{1}{(h-1)!} & \frac{1}{(h-2)!} & \cdots & \frac{1}{1!} & 1
\end{pmatrix} = e^{\alpha_i t} K_h.
$$

Then the formula for the exponential one of the matrix $\mathbf{A}$, is:

$$
e^{\mathbf{A}t} = Pdiag\left(e^{\alpha_1 t}K_{h_1}, e^{\alpha_2 t}K_{h_2}, \cdots, e^{\alpha_m t}K_{h_m}\right)P^{-1}
$$

This formula allows to simplify the matrix $\mathbf{B}\left(e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{B}ds\right)'$, and the matrix $e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{B}ds\mathbf{B}'$:

$$
\begin{aligned}
& e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{B}ds\mathbf{B}' \\
= \;& P\left(I - diag\left(-\frac{1}{\alpha_1}e^{-\alpha_1 t}K_{h_1}^2, ..., -\frac{1}{\alpha_m}e^{-\alpha_m t}K_{h_m}^2\right)\right)P^{-1}\mathbf{B}\mathbf{B}'
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{B}\left(e^{\mathbf{A}t}\int_{t_0}^{t}e^{-\mathbf{A}s}\mathbf{B}ds\right)' \\
= \ &\mathbf{B}\left[P\left[\begin{array}{c} diag\left(-\frac{1}{\alpha_1}K_{h_1}^2,...,-\frac{1}{\alpha_m}K_{h_m}^2\right) \\ +diag\left(-\frac{1}{\alpha_1}e^{-\alpha_1 t}K_{h_1}^2,...,-\frac{1}{\alpha_m}e^{-\alpha_m t}K_{h_m}^2\right)\end{array}\right]P^{-1}\mathbf{B}\right]'
\end{aligned}
$$

The components of these matrices have the form $\sum\limits_k c_k e^{t\alpha_k}$.

Also the terms $\mathbf{P}\left(e^{\mathbf{A}t}\zeta_0\right)'$, $\mathbf{P}'e^{\mathbf{A}t}\zeta_0$ can be simplified, leaving vectors with components of the form $\sum\limits_k c_k e^{t\alpha_k}$.

On the other hand, the previous system can be rewriten in vectorial notation doing the correspondence:

$$
\left(\begin{array}{ccc} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn}\end{array}\right) \rightarrow \left(\begin{array}{ccccccc} c_{11} & \cdots & c_{1d} & ,\cdots, & c_{d1} & \cdots & c_{dd}\end{array}\right);
$$

If the identification $\rho = \left(\begin{array}{ccccccc} \rho_{11} & \cdots & \rho_{1d} & ,\cdots, & \rho_{d1} & \cdots & \rho_{dd}\end{array}\right)$ is used and if we consider the Jordan's canonical form of the matrix $\mathbf{A}$, seen previously, then the previous dynamical system will be written as:

$$
\begin{aligned}
\dot{\rho} &= \left(\mathbf{a}_{ij}I + \delta_{ij}\left(\mathbf{A}+E_{ij}\right)\right)\rho + V(t) + V, \\
i,j &= 1,2,...,d;
\end{aligned}
$$

where $\mathbf{A} = (\mathbf{a}_{ij})$, the vector V is a vector of constant components that corresponds to the simplification of the terms with the integral and $V(t)$ is a vector with combinations of terms of the form $\sum\limits_k c_k e^{t\alpha_k}$, that corresponds to the simplification of the previous terms that do not have the integral. When the eigenvalues α_i have a negative real part, the following dynamical system can be associated to the previous dynamical system:

$$
\begin{aligned}
\dot{\rho} &= \left(\mathbf{a}_{ij}I + \delta_{ij}\left(\mathbf{A}+E_{ij}\right)\right)\rho, \\
i,j &= 1,2,...,d.
\end{aligned}
$$

These dynamical systems are equivalent, because the norm of the difference between the respective vectorial fields is $\|V(t)\|$; this norm is so that $\|V(t)\| \rightarrow 0$, when $t \rightarrow \infty$. Finally, the dynamic in the mean-square is based in the eigenvalues of the matrix of $d^2 \times d^2$,

$$
\begin{aligned}
&\left(\mathbf{a}_{ij}I + \delta_{ij}\left(\mathbf{A}+E_{ij}\right)\right), \\
i,j &= 1,2,...,d;
\end{aligned} \tag{5}
$$

because the vector V is of constant components.

4. The algorithm for the reduction of $(\mathbf{a}_{ij} I + \delta_{ij} (\mathbf{A} + E_{ij}))$.

Consider

$$\beta_{d^2} \doteq (a_{ij} I_d + \delta_{ij} (\mathbf{A} + E_{ij}))_{d^2};$$

the symbol $\doteq$ denotes "is defined", d, i, j are positive integers. And:

$\rho = (\rho_{11}, \cdots, \rho_{1d^2}, \rho_{21}, \cdots, \rho_{2d^2}, \cdots, \rho_{d1}, ..., \rho_{d^2d^2})$,

$\rho_{ij} \doteq \rho_{ij}(t)$ are time dependent functions of real values, t real number

$\dot{\rho}$ is the derivative of ρ with respect to time; all the variables with a point above indicate the same concept,

$\mathbf{A}_d = (a_{ij})_{d\times d}$ where $a_{ij} \in R$,

I_d is the $d \times d$ identity matrix,

δ_{ij} is the Kronecker's delta and

E_{ij} is a $d \times d$ matrix defined as following

$$E_{ij} = \begin{cases} E_{ii} & \text{if } i = j \\ (0)_{d\times d} & \text{if } i \neq j \end{cases}$$

where E_{ii} is the $d \times d$ matrix with ones on the entry (i, i) and zeros elsewhere.

When the order $n \times n$ (n, a positive integer) of a square matrix is indicated we will write A_n.

As an important condition it is assumed that the state variables satisfy

$$\rho_{ij} = \rho_{ji}.$$

What mainly concerns are the eigenvalues of the matrix β_{d^2}.

The matrix β_{d^2} contains d^2 eigenvalues, of which only $T_d \doteq \frac{d(d+1)}{2}$ are remarkable because are those that indicate the stability of a dynamical system in square mean.The triangular numbers are formed by partial sum of the series $1 + 2 + ... + n$. So $T_n = \frac{n(n+1)}{2}$. In other words triangular numbers form the series $1, 3, 6, 10, 15, 21, 28...$ and receive this name because they are the number of dots needed to make successive triangular arrays of dots. These T_d numbers correspond to the matrix which will be called α_{T_d}.

Here is proposed a transformation algorithm of the matrix β_{d^2} to the matrix α_{T_d}.

Such algorithm operates on rows and columns of the matrix β_{d^2}. In order to obtain a better undestanding of the operations contained in the algorithm enunciated later, let us number the columns of β_{d^2} with simple indices $1, 2, 3, ..., d^2$; or renumber them with double indices

$$11, 12, 13.., 1d, 21, 22, ..., 2d, ..., d1, d2, ..., dd.$$

The ordered pair (i, j) corresponds to the double index ij.

When applying the algorithm, simple indices and double indices are used indistinctly. So, we propose the next bijective function. Let $\mathfrak{T}_d = \{1, 2, ..., d\}$, then

$$\begin{aligned} f &: \mathfrak{T}_d \times \mathfrak{T}_d \to \mathfrak{T}_{d^2}, \\ f(i,j) &= j + (i-1)d. \end{aligned}$$

The inverse function was obtained solving the Diophantine equation $k = j + (i-1)d$ for i and j as functions of k;

$$\begin{array}{rcl} f^{-1} & : & \mathfrak{T}_{d^2} \to \mathfrak{T}_d \times \mathfrak{T}_d, \\ f^{-1}(k) & = & \left(-d-k-\left[1-\frac{(d+k)(d+1)}{d}\right], \ (d+k)(d+1)+d\left[1-\frac{(d+k)(d+1)}{d}\right] \right), \end{array}$$

where $[x]$ is the greatest integer function. For details of the function $[x]$, see [8, Greatest Integer Function, p. 180].

In order to apply the algorithm we are now ready to introduce the following definition.

Definition 1 *Two double indices ij, kl are equivalent if $i = l$ and $j = k$.*

Algorithm A The instructions for the transformation of the matrix β_{d^2} to the matrix α_{T_d} are:

$A1$. Add the columns with equivalent indices.
$A2$. Replace a column by the obtained sum and eliminate the other column.
$A3$. Suppress all the rows that are equal to one given, except this row.
We use the following notations:

$$\begin{array}{c} \textit{sum}\,\mathrm{col}(j,k), \\ \textit{sum}\,\mathrm{row}(j,k), \end{array}$$

It means that we add the columns (rows) j, k and the result will be replaced in column (row) j;

$$\begin{array}{c} \textit{int}\,\mathrm{col}(i,j), \\ \textit{int}\,\mathrm{row}(i,j), \end{array}$$

It means that columns (rows) i and j will be interchanged.

In the following subsections parts i, ii and iii will be proved.

Remark 1*:*

The eigenvalues of α_{T_d} are contained in the set of the eigenvalues of β_{d^2}. Hence, it suffices to show that for some matrix $\gamma_{T_{d-1}}$

$$p_{\beta_{d^2}}(\lambda) = p_{\alpha_{T_d}}(\lambda)\, p_{\gamma_{T_{d-1}}}(\lambda),$$

where

$$\begin{array}{rcl} p_{\beta_{d^2}}(\lambda) & = & \det\left(\beta_{d^2} - \lambda I_{d^2}\right), \\ p_{\alpha_{T_d}}(\lambda) & = & \det\left(\alpha_{T_d} - \lambda I_{T_d}\right), \\ p_{\gamma_{T_{d-1}}}(\lambda) & = & \det\left(\gamma_{T_{d-1}} - \lambda I_{T_{d-1}}\right). \end{array}$$

First consider applying the operations related to the $\lambda-matrices$ to the matrix $\beta_{d^2} - \lambda I_{d^2}$, see [7, División de λ-matrices, p. 170], with the purpose of obtaining the following decomposition

$$B_{d^2} \doteq \left(\begin{array}{ccc|c} & \alpha_{T_d} - \lambda I_{T_d} & & \varepsilon_{T_d \times T_{d-1}} \\ 0 & \dots & 0 & \\ \vdots & & \vdots & \gamma_{T_{d-1}} - \lambda I_{T_{d-1}} \\ 0 & \dots & 0 & \end{array} \right), \tag{6}$$

where $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$ is a $T_{d-1} \times T_d$ matrix.

Then we calculate the determinant of B_{d^2}.

As the determinant does not change when applying $\lambda - matrices$ transformations and by a well-known property of the determinants (see [5, Produkt zweier Determinanten, p. 60]) it follows:

$$\det B_{d^2} = p_{\beta_{d^2}}(\lambda) = p_{\alpha_{T_d}}(\lambda)\, p_{\gamma_{T_{d-1}}}(\lambda).$$

The following procedures are constructed in order to obtain the B_{d^2} decomposition.

The matrix $\alpha_{T_d} - \lambda I_{T_d}$ is obtained by procedures 1 and 2.

Procedure 1. T_{d-1} instructions. This procedure adds the columns with equivalent indices.

For $i = 1, 2, ..., d-1$ y $j = i+1, ..., d$,

$$sum\, \mathrm{col}\, (f(i,j), f(j,i)).$$

Procedure 2. $2T_{d-1}$ instructions. This procedure moves the repeated columns towards right and the repeated rows downwards.

For $k = 1, ..., d-1$.

$$\begin{array}{lll} \text{If} & k = 1 & i = (d+1), ..., (2d-1), \\ & k = 2 & i = 2d, ..., (3d-3), \\ & k = 3, ..., d-2 & i = kd + 1 - T_{k-1}, ..., (2d-k)(k+1), \\ & k = d-1 & i = d(k+1) - T_{k-1}. \end{array}$$

Hence,

$$\begin{gathered} int\, \mathrm{row}\, (i, i+T_k), \\ int\, \mathrm{col}\, (i, i+T_k). \end{gathered}$$

Remark 2 *When interchanging a row we must interchange the column (the interchanges are noncommutative).*

Procedure 3. T_{d-1} instructions. With this procedure we obtain the zero submatrix from the row T_d+1 to the row d^2 and it consists of two parts.

First Part.$T_{d-1}-T_{\left[\frac{d}{2}\right]}$ instructions. Only a part of the required zero submatrix is obtained. The following formula gives us the pair of rows that will be subtracted

For $l=1,...,\left[\frac{d-1}{2}\right]$ and $k=0,...,d-l-1$,

$$sum\,\mathrm{row}\left((d-l)(d+1)-k,-T_d+T_l+T_k+kl\right). \tag{7}$$

Second Part. $T_{\left[\frac{d}{2}\right]}$ instructions. The other part of the zero submatrix is obtained. In the first part the method was direct; now the idea for completing the zero matrix is to divide this second part in the following seven steps.

Step 1. With M_d we obtain a list of rows from α_{T_d} that will be subtracted later with the rows (that have not been made zero) of the submatrix that goes from the row T_d+1 to the row d^2. Let $M_d:\Sigma_l\times\Sigma_k\to\Sigma_{M_d}$ where $\Sigma_l=\left\{\left[\frac{d+1}{2}\right],...,d-1\right\}$, $\Sigma_k=\left\{l-\left[\frac{d+1}{2}\right]+1,...,\left[\frac{d}{2}\right]\right\}$ and $\Sigma_{M_d}\subset\{2,3,...,T_d\}$,

$$M_d(l,k)=-T_{\left[\frac{d}{2}\right]}-T_{k-1}+d\left(\left[\frac{d}{2}\right]+1-k\right)-l+1. \tag{8}$$

Notation 1*Let ϕ_μ be the μ^{th} element of the list Φ, where Φ contains the elements of range M_d.*

Step 2. L_d calculate in ascending order the rows that have not been made zero; these rows are contained in the submatrix that goes from the row T_d+1 to the row d^2. Let $L_d:\Sigma_i\times\Sigma_j\to\Sigma_{L_d}$ where $\Sigma_i=\left\{1,...,\left[\frac{d}{2}\right]\right\}$, $\Sigma_j=\left\{0,1,...,\left[\frac{d}{2}\right]-i\right\}$ and $\Sigma_{L_d}\subset\left\{T_d+1,...,d^2\right\}$,

$$L_d(i,j)=d^2-\left(\left[\frac{d}{2}\right]-i\right)(d+1)+j. \tag{9}$$

Step 3. Renumber the elements of Σ_{L_d} in a simple list from 1 to $T_{\left[\frac{d}{2}\right]}$. Let $R_d:\Sigma_i\times\Sigma_j\to\Sigma_{R_d}$ where $\Sigma_{R_d}=\left\{1,...,T_{\left[\frac{d}{2}\right]}\right\}$,

$$R_d(i,j)=i+j+(i-1)\left[\frac{d}{2}\right]-T_{i-1}. \tag{10}$$

It can be noted that formulae 7, 8, 9 and 10 always take integer values.

The list $1,...,T_{\left[\frac{d}{2}\right]}$, that corresponds with the range R_d, is not in the needed order. To find this order we will consider the permutation σ which belongs to the symmetrical group $\mathfrak{S}_{T_{\left[\frac{d}{2}\right]}}$. Thus, σ is defined as

$$\sigma=\begin{pmatrix} 1,2,...,T_{\left[\frac{d}{2}\right]} \\ S_1\mid S_2\mid ...\mid S_{\left[\frac{d}{2}\right]}\end{pmatrix}.$$

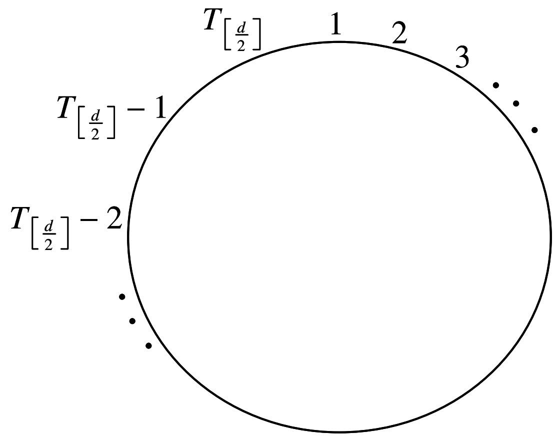

Figure 1: Circular arrangement

Where the partitions S_ν, that consist of $\left[\frac{d}{2}\right]-\nu+1$ elements, are generated as follows.

First the partition S_1 is generated with the formula:

$$\frac{\left[\frac{d}{2}\right]\left(\left[\frac{d}{2}\right]+(-1)^d\right)}{2}-\left\lceil\frac{2\left[\frac{d}{2}\right]}{d}\right\rceil-h, \tag{11}$$

where $h=1,...,\left[\frac{d}{2}\right]$ and this can be written as:

$$S_1=\begin{pmatrix} \frac{\left[\frac{d}{2}\right]\left(\left[\frac{d}{2}\right]+(-1)^d\right)}{2}-\left\lceil\frac{2\left[\frac{d}{2}\right]}{d}\right\rceil+1-0 \\ \frac{\left[\frac{d}{2}\right]\left(\left[\frac{d}{2}\right]+(-1)^d\right)}{2}-\left\lceil\frac{2\left[\frac{d}{2}\right]}{d}\right\rceil+1-1 \\ \vdots \\ \frac{\left[\frac{d}{2}\right]\left(\left[\frac{d}{2}\right]+(-1)^d\right)}{2}-\left\lceil\frac{2\left[\frac{d}{2}\right]}{d}\right\rceil+1-\left(\left[\frac{d}{2}\right]-1\right) \end{pmatrix}.$$

To generate the partitions S_2 to $S_{\left[\frac{d}{2}\right]}$ we may define a circular arrangement with the numbers from 1 to $T_{\left[\frac{d}{2}\right]}$ written in clockwise direction, see figure 1.

The elements of the partition S_1, generated with formula 11, are selected in the circular adjustment. Once selected, they must not be considered again in the following steps of the algorithm.

Now we must apply Algorithm S to generate the partitions from S_2 to $S_{\left[\frac{d}{2}\right]}$ with $t=1,...,\left[\frac{d}{2}\right]-1$, respectively.

Algorithm S Let $t=1,...,\left[\frac{d}{2}\right]-1$.

B1. Select the minimum t that have not been used and the partition S_{t+1} will be generated. The selection is always made in anticlockwise direction.

B2. Locate in the last element of the partition S_{t+1}.

B3. Obtain the remainder $r_{(t,d)}$, which corresponds to the numbers we must advance in the circular arengment. This number was obtained applying the division algorithm to $d-\left[\frac{d}{2}\right]+t+1$ and $T_{\left[\frac{d}{2}\right]-t}$.

$$r_{(t,d)} = d-\left[\frac{d}{2}\right]+t+1-\left[\frac{d-\left[\frac{d}{2}\right]+t+1}{T_{\left[\frac{d}{2}\right]-t}}\right]\left(T_{\left[\frac{d}{2}\right]-t}\right).$$

It can be noted that the divisor $T_{\left[\frac{d}{2}\right]-t}$ never takes the zero value.

B4. If $\begin{cases} r_{(t,d)} > 0 & \text{we advance } r_{(t,d)} \text{ elements} \\ r_{(t,d)} = 0 & \text{we advance } T_{\left[\frac{d}{2}\right]-t} \text{ elements} \end{cases}$

We have to advance also in anticlockwise direction.

Remark 3 *When* $t=\left[\frac{d}{2}\right]-1$ *we always obtain* $r_{\left(\left[\frac{d}{2}\right]-1,d\right)}=0$.

B5. Select $\left[\frac{d}{2}\right]-t$ elements including the element at which we arrived in **B4**,

B6. The elements selected in **B5** form the partition S_{t+1}, and they can be written in a list conserving the order in which they were selected:

$$S_{t+1} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}.$$

Remark 4 *It could happen that the elements of a partition are not successive because when we advance we could found selected elements that must not be considered.*

B7. Return to **B1**.

Step 4. Write the partitions S_ν as follows.

$$\Lambda = \begin{pmatrix} S_1 \\ \vdots \\ S_{\left[\frac{d}{2}\right]-1} \\ S_{\left[\frac{d}{2}\right]} \end{pmatrix}.$$

Notation 2 *Let* λ_μ *be the* μ^{th} *element of the list* Λ, $\mu=\left\{1,2,...,T_{\left[\frac{d}{2}\right]}\right\}$.

Step 5. Evaluate the inverse of R_d. More precisely, we obtain the i's and j's that solve $R_d(i,j)=w$, with w an element of the list Λ. When i and j are extended to real numbers, R_d is a parabola for each value of w. In order to calculate the inverse function R_d^{-1}, we found all the points (x,y) with rational coordinates that belong to the parabola R_d, see the methods in [8, Rational points on curves, p. 249].

We may consider the point $\left(0, w+\left[\frac{d}{2}\right]\right)$ with integer coordinates in R_d and obtain all the lines with slope m that intersect that point; m is obtained according to the values of w. Finally we calculate the intersection of this lines with the parabolas R_d.

With the same previous notations, let $R_d^{-1}: \Sigma_{R_d} \to \Sigma_i \times \Sigma_j$,

$$R_d^{-1}(w) = \left(2\left(m+\left[\frac{d}{2}\right]+\frac{3}{2}\right),\ 2m\left(m+\left[\frac{d}{2}\right]+\frac{3}{2}\right)+w+\left[\frac{d}{2}\right]\right), \tag{12}$$

where $m = \left\{\frac{-2\left[\frac{d}{2}\right]-2}{2}, \frac{-2\left[\frac{d}{2}\right]-1}{2}, \frac{-2\left[\frac{d}{2}\right]}{2}, ..., \frac{-\left[\frac{d}{2}\right]-3}{2}\right\}$ and will be assigned to the w as follows:

$$\begin{array}{ll} w = 1, ..., \left[\frac{d}{2}\right] & m = \frac{-2\left[\frac{d}{2}\right]-2}{2} \\ w = \left[\frac{d}{2}\right]+1, ..., 2\left[\frac{d}{2}\right]-1 & m = \frac{-2\left[\frac{d}{2}\right]-1}{2} \\ w = 2\left[\frac{d}{2}\right], ..., 3\left[\frac{d}{2}\right]-3 & m = \frac{-2\left[\frac{d}{2}\right]}{2} \\ \vdots & \vdots \\ w = T_{\left[\frac{d}{2}\right]} & m = \frac{-\left[\frac{d}{2}\right]-3}{2} \end{array}$$

Step 6. Here we obtain the rows that have not been made zero in the order of Λ. For this, we calculate the composite function $L_d \circ R_d^{-1}: \Sigma_{R_d} \to \Sigma_{L_d}$,

$$\left(L_d \circ R_d^{-1}\right)(\lambda_\mu) = L_d\left(R_d^{-1}(\lambda_\mu)\right) = L_d(i(\lambda_\mu), j(\lambda_\mu)).$$

Step 7. Finally we have:

$$sum\,\text{row}\left(\left(L_d \circ R_d^{-1}\right)(\lambda_\mu), \phi_\mu\right). \tag{13}$$

Lemma 1 allows us to extend the validity of these results from order d to order $d+1$, see section on lemma 1.

Once procedures 1, 2 and 3 are applied, the typical elements for the matrices a_{T_d} and $\gamma_{T_{d-1}}$ are deduced as the following descriptions.

4.0.1. Description of α_{T_d}

1. In the diagonal we have elements of type $a_{ii}+a_{jj}$ and $2a_{ii}+1$, with $i, j = 1, ..., d$, $i < j$.
2. The elements that do not belong to the diagonal can be written as $\alpha_{kl} = \varepsilon a_{rs}$. The element α_{kl} correspons with the row k and the column l.

Case 1*If in the row k we find the diagonal element $a_{\widehat{i}\widehat{i}} + a_{\widehat{j}\widehat{j}}$ and in the column l, the diagonal element $a_{\overline{i}\overline{i}} + a_{\overline{j}\overline{j}}$, ε, r and s are defined as:*

$$If \begin{cases} \widehat{i} = \overline{i} & \Rightarrow \quad \varepsilon = 1, \quad r = \widehat{j}, \quad s = \overline{j} \\ \widehat{j} = \overline{j} & \Rightarrow \quad \varepsilon = 1, \quad r = \widehat{i}, \quad s = \overline{i} \\ \widehat{i} = \overline{j} & \Rightarrow \quad \varepsilon = 1, \quad r = \widehat{j}, \quad s = \overline{i} \\ \widehat{j} = \overline{i} & \Rightarrow \quad \varepsilon = 1, \quad r = \widehat{i}, \quad s = \overline{j} \\ otherwise & \Rightarrow \quad \varepsilon = 0 \end{cases}$$

Case 2 *If in the row k we find the diagonal element $a_{\widehat{i}\widehat{i}}+a_{\widehat{j}\widehat{j}}$ and in the column l, the diagonal element $2a_{\overline{i}\overline{i}}+1$, ε, r and s are defined as:*

$$\text{If}\begin{cases} \widehat{i}=\overline{i} & \Rightarrow \varepsilon=1, \quad r=\widehat{j}, \quad s=\overline{i} \\ \widehat{j}=\overline{i} & \Rightarrow \varepsilon=1, \quad r=\widehat{i}, \quad s=\overline{i} \\ \textit{otherwise} & \Rightarrow \varepsilon=0 \end{cases}$$

Case 3 *If in the row k we find the diagonal element $2a_{\widehat{i}\widehat{i}}+1$ and in the column l, the diagonal element $a_{\overline{i}\overline{i}}+a_{\overline{j}\overline{j}}$, ε, r and s are defined as:*

$$\text{If}\begin{cases} \widehat{i}=\overline{i} & \Rightarrow \varepsilon=2, \quad r=\widehat{i}, \quad s=\overline{j} \\ \widehat{i}=\overline{j} & \Rightarrow \varepsilon=2, \quad r=\widehat{i}, \quad s=\overline{i} \\ \textit{otherwise} & \Rightarrow \varepsilon=0 \end{cases}$$

Case 4 *If in the row k we find the diagonal element $2a_{\widehat{i}\widehat{i}}+1$ and in the column l, the diagonal element $2a_{\overline{i}\overline{i}}+1$ then $\varepsilon=0$.*

4.0.2. Description of $\gamma_{T_{d-1}}$

1. In the diagonal we have elements of type $a_{ii}+a_{jj}$, with $i,j=1,...,d$, $i<j$.
2. The elements that do not belong to the diagonal can be written as $\gamma_{kl}=\varepsilon a_{rs}$. The element γ_{kl} correspons with the row k and the column l. In the row k we find the diagonal element $a_{\widehat{i}\widehat{i}}+a_{\widehat{j}\widehat{j}}$ and in the column l, the diagonal element $a_{\overline{i}\overline{i}}+a_{\overline{j}\overline{j}}$. With this diagonal elements ε, r and s are defined as follows:

$$\text{If}\begin{cases} \widehat{i}=\overline{i} & \Rightarrow \varepsilon=1, \quad r=\widehat{j}, \quad s=\overline{j} \\ \widehat{j}=\overline{j} & \Rightarrow \varepsilon=1, \quad r=\widehat{i}, \quad s=\overline{i} \\ \widehat{i}=\overline{j} & \Rightarrow \varepsilon=-1, \quad r=\widehat{j}, \quad s=\overline{i} \\ \widehat{j}=\overline{i} & \Rightarrow \varepsilon=-1, \quad r=\widehat{i}, \quad s=\overline{j} \\ \text{otherwise} & \Rightarrow \varepsilon=0 \end{cases}$$

With the previous descriptions, different matrices a_{T_d} and $\gamma_{T_{d-1}}$ can be obtained; the difference depends on the order of their diagonal elements. However, these matrices always will have the same characteristic polynomial.

5. Proof of Lemma 1

We make use of the recursion theorem.[4, Inducción y recursión p. 58].

Lemma 1 *The procedures 1, 2 and 3 transforms β_{d^2} in the matrix B_{d^2}, with $d=2,3,...$*

Proof. By the recursion theorem we have $U=\{(m_{ij})_{m\times n}\mid i=1,2,...,m \text{ and } j=1,2,...,n\}$ the set of all the matrices of all the orders and $\mathfrak{B}=\{\beta_4\}$ the basis set.

Let $g:U\to U$ such that $g(\beta_{d^2})=\beta_{(d+1)^2}$ and let $C=\{\beta_{d^2}\mid d=2,3,...\}$ the freely generated set beginning from $\mathfrak{B}$ by g_C.

Suppose that $V = \{B_{d^2} \mid d = 2, 3...\}$, and the functions G and h are such that $h : \mathfrak{B} \to V$ such that $h(\beta_4) = B_4$ and
$G : V \to V$ such that $G(B_{d^2}) = B_{(d+1)^2}$
by hypothesis the procedures $1, 2,$ and 3 allow us to make this definition of h.
Then exist only one function $\overline{h} : C \to V$ such that:
i) For all $x \in \mathfrak{B}$, $\overline{h}(x) = h(x)$.
ii) And for $x \in C$, $\overline{h}(g(x)) = G\left(\overline{h}(x)\right)$.
That function is $\overline{h}(\beta_{d^2}) = B_{d^2}$ and for the same reason we define h, we can to define $\overline{h}$ too. ■

6. Proof of Lemma 2

To prove Lemma 2 (see next lines) we propose a transformation of $\gamma_{T_{d-1}}$ such transformation will be call Algorithm S-C.

We will transform $\gamma_{T_{d-1}}$ in $\gamma^0_{T_{d-1}}$ where $\gamma^0_{T_{d-1}}$ will be a equivalent description in the sense that we only can make interchanges of rows and columns; however, these matrices always will have the same characteristic polynomial. For this transformation we propose the next diagonal order of the elements. Let $a_{ii} + a_{jj}$ with $i = 2, ..., d$ and $j = 1, ..., i-1$.

To make this transformation we must follow the interchanges of rows and columns implicit in the next algorithm who is similar to Algorithm S defined on page 12.

Algorithm S-C

Step 1. Let $1, ..., T_{d-1}$ the elements who will be consider in the permutation σ which belongs to the symmetrical group $\mathfrak{S}_{T_{d-1}}$. Thus, σ is defined as

$$\sigma = \begin{pmatrix} 1, 2, ..., T_{d-1} \\ S_1 \mid S_2 \mid ... \mid S_{d-1} \end{pmatrix}.$$

The partitions S_ν consist of $d - \nu$ elements and they are genereted with the next steps.

Step 2. We may define a circular arrangement with the numbers from 1 to T_{d-1} written in clockwise direction, see figure 7.

Step 3. To generate partition S_1 we select $d-1$ elements from T_{d-1}. The selection is always made in anticlockwise direction and once selected, they must not be considered again in the following steps of the algorithm.

$$S_1 = \begin{pmatrix} T_{d-1} - 1 \\ T_{d-1} - 2 \\ \vdots \\ T_{d-1} - (d-1) \end{pmatrix}.$$

Step 4. Let $t = 2, ..., d-1$.

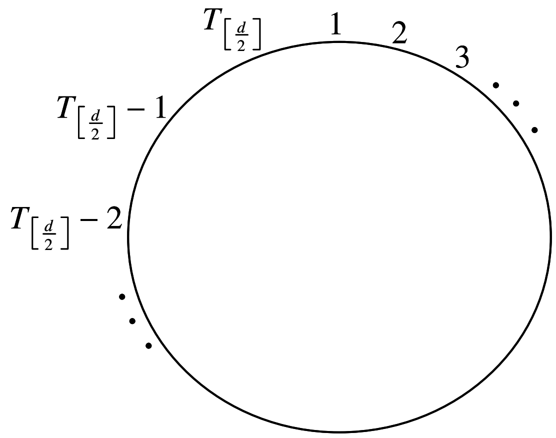

Figure 2: Circular arrangement

A1.Select the minimum t that have not been used and the partition S_t will be generated.

A2. Locate in the last element of the partition S_t.

A3.We must advance $t+1$ places in the circular arengment. We have to advance also in anticlockwise direction.

A4.Select $d-t$ elements including the element at which we arrived in **A3**,

A5.The elements selected in **A4** form the partition S_t, and they can be written in a list conserving the order in which they were selected:

$$S_t = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}.$$

Remark 5*It could happen that the elements of a partition are not successive because when we advance we could found selected elements that must not be considered.*

A6. Return to **A1**.

Step 5. Now we have the list of elements $1, ..., T_{d-1}$ but with other order who is generated with the blocks S_ν and write the partitions S_ν as follows:

$$U = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_{d-1} \end{pmatrix}.$$

Notation 3*Let u_μ be the μ^{th} element of the list U, with $\mu = 1, 2, ..., T_{d-1}$.*

Step 6. We have another list

$$V = \begin{pmatrix} T_{d-1} \\ \vdots \\ 2 \\ 1 \end{pmatrix}.$$

Notation 4 *Let v_μ be the μ^{th} element of the list V, with $\mu = 1, 2, ..., T_{d-1}$.*

Step 7. Finally to have $\gamma^0_{T_{d-1}}$, the interchanges are:

$$\begin{aligned} &int\,\mathrm{row}(u_\mu, v_\mu), \\ &int\,\mathrm{col}(u_\mu, v_\mu), \end{aligned}$$

with $\mu = 1, 2, ..., T_{d-1}$.

Lemma 2 *Let us consider* $B_{d^2} = \begin{pmatrix} \alpha_{T_d} - \lambda I_{\frac{d(d+1)}{2}} & & \varepsilon \\ 0 & \dots & 0 & \\ \vdots & & \vdots & \gamma_{T_{d-1}} - \lambda I_{\frac{d(d-1)}{2}} \\ 0 & \dots & 0 & \end{pmatrix}$. *There exist a description of matrix $\gamma_{T_{d-1}}$ which we can interpret as a function who transforms $\gamma_{T_{d-1}}$ in $\gamma^0_{T_{d-1}}$, then this description also transforms γ_{T_d} in $\gamma^0_{T_d}$.*

Proof. To prove that Algorithm S-C transforms γ_{T_d} in $\gamma^0_{T_d}$ it is enough to consider:

First of all that Algorithm S-C transforms $\gamma_{T_{d-1}}$ in $\gamma^0_{T_{d-1}}$.

And then we observe, that the components of $\gamma^0_{T_{d-1}}$ are contained on the components of γ_{T_d}, therefore if we can obtain γ_{T_d}, by iterchanging rows and columns, the matrix will be as follows

$$\begin{pmatrix} \gamma^0_{T_{d-1}} & * \\ * & \delta \end{pmatrix},$$

where δ is some matrix, and we must have $\gamma^0_{T_d}$. Let us note that the interchangings of rows and columns must be done to γ_{T_d} are the same if we applied Algorithm S-C to such matrix.

Hence, Algorithm S-C is a particular case of the descriptions set of matrix $\gamma_{T_{d-1}}$ such transforms γ_{T_d} in $\gamma^0_{T_d}$. ■

7. References

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