On The Root Clustering of Matrices

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Abstract

Root clustering problems of matrices are considered. Here we are given conditions for eigenvalues of a matrix to lie in a prescribed subregion $\mathbb{D}$ of the complex plane. The region $\mathbb{D}$ (stability region) is defined by rational functions. A simple necessary and sufficient condition for stability of a single matrix is obtained. For a commutting polynomial family a necessary and sufficient condition in terms of a common solution to a set of Lyapunov inequalities is derived. A simple sufficient condition for the existence of a common solution for a commutting quadratic polynomial matrix family is given. A sufficient condition for the existence of a common solution to the Lyapunov inequalities for two 3×3 dimensional z-matrices is also given.

Key words: Stability region, Lyapunov inequality, Common solution ,Commuting family, Z-matrices

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1 Introduction

Let $\mathbb{R}^n$ be the set of real n vectors, $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) be the set of $n \times n$ real (complex) matrices. For $P \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) the symbol $P > 0$ means that P is symmetric (Hermitian) and positive definite. Let the subregion $\mathbb{D}$ of the complex plane $\mathbb{C}$ be defined as

$$\mathbb{D} = \{z \in \mathbb{C} : Re f_j(z)\bar{g}_j(z) < 0, j = 1, 2, \ldots, m\}, \tag{1}$$

where f_j and g_j are polynomials with real coefficients, and $\bar{g}$ is the complex conjugate of g. The inequality $Re f_j(z)\bar{g}_j(z) < 0$ is equivalent form of the inequality $Re\ r_j(z) < 0$, where $r_j(z) = \frac{f_j(z)}{g_j(z)}$.

The region $\mathbb{D}$ defined in (1) will also be referred as the stability region. It is a generalization of the known stability regions:

If m=1, f(z)=z, g(z)=1 it is Hurwitz stability region,

If m=1, f(z)=z+1, g(z)=z-1 it is Schur stability region,

If m=2, f_1(z)=z, f_2(z)=-z^2, g_j(z)=1 ($j = 1, 2$) it is $\frac{\pi}{4}$ left sector stability region,

If m=2, f_1(z)=z+a, f_2(z)=-z-b, g_1(z)=z-a, g_2(z)=z-b

it is the ring $\{z \in \mathbb{C} : b < |z| < a\}$.

By the Lyapunov theorem the matrix $A \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) is Hurwitz stable if and only if there exists $P \in \mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) , $P > 0$ such that

$$\begin{gathered} A^T P + PA < 0 \\ (A^* P + PA < 0) \end{gathered} \tag{2}$$

where A^T (A^*) denotes the transpose (conjugate transpose) of A.

In [1] the following result is obtained (see [1], Theorem 1).

Theorem 1 *([1]). Let the stability region* Ω *be defined as*

$$\Omega = \{z \in \mathbb{C} : Ref_j(z) < 0, j = 1, 2, \dots, m\}, \tag{3}$$

where $f_j (j = 1, 2, \dots, m)$ *are all polynomials. Then the matrix* $A \in \mathbb{R}^{n\times n}$ *is* Ω *-stable if and only if there exists a matrix* $P \in \mathbb{R}^{n\times n}$, $P > 0$ *such that for all* $j = 1, 2, \dots, m$

$$[f_j(A)]^T P + P [f_j(A)] < 0. \tag{4}$$

In [2], the following result on the existence of a common $P > 0$ for commuting matrices $A_1, A_2, \dots, A_k$ is given (see [2], Theorem 2).

Theorem 2 . *Let* $A_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2, \dots, k)$ *be Hurwitz stable and commute pairwise. Then there exists* $P \in \mathbb{R}^{n\times n}$, $P > 0$ *such that for all* $i = 1, 2, \dots, k$

$$A_i^T P + P A_i < 0. \tag{5}$$

Note that in [2] an explicit method of generating a common P is also presented (see (13) below).

In this work by using Theorems 1 and 2 we prove a simple criterion for $\mathbb{D}$-stability of a matrix A. We show that $\mathbb{D}$-stability of a matrix A is equivalent to the Hurwitz stability of the matrices $f_1(A)g_1^{-1}(A), \dots, f_m(A)g_m^{-1}(A)$ (Theorem 7).

For $t \in [0,1]$ and $A_i \in \mathbb{C}^{n\times n}$ $(i = 1, 2, \ldots, m)$ define

$$A(t) = A_0 + tA_1 + t^2 A_2 + \ldots + t^k A_k, \tag{6}$$

$$\mathcal{A} = \{A(t) : t \in \ [0,1]\}\,. \tag{7}$$

In [3], for $A_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2, \ldots, k)$, using the guardian map concept, the Hurwitz stability problem for the family $\mathcal{A}$ in (7) is considered and a condition for stability is derived . In this work for the commuting family (6), (7), i.e. in the case of $A_i A_j = A_j A_i$ $(i, j = 1, 2, \ldots, k)$, we give a necessary and sufficient condition for $\mathbb{D}$-stability in terms of a common solution to a set of Lyapunov inequalities (Theorem 11). For the case of quadratic family (i.e. k=2 in (6)) sufficient conditions for the existence of a common solution to set of Lyapunov inequalities are obtained (Theorems 13 and 15).

Finally, a sufficient condition for the existence of a common solution for a pair of z-matrices is given (Theorem 17).

2 Stability of a single matrix

In this section we give a criterion for the $\mathbb{D}$-stability of a matrix $A \in \mathbb{R}^{n\times n}$.

Lemma 3 . *Let $f(z)$ and $g(z)$ be polynomials, $A \in \mathbb{R}^{n\times n}$. If $g(A)$ is invertible then $f(A)$ and $g^{-1}(A)$ commute.*

The proof follows from the equality $f(A)g(A) = g(A)f(A)$.

Lemma 4 . *Let $f_j(z)$, $g_j(z)$ $(j = 1, 2, \ldots, m)$ are polynomials and $g_j(A)$ are invertible for all $j = 1, 2, \ldots, m$. Then the matrices $r_j(A) = f_j(A).\ g_j^{-1}(A)$*

$(j = 1, 2, \ldots, m)$ *are commutative.*

Proof. By Lemma 3 the following is true

$$[g_j(A)g_i(A)]^{-1} f_i(A)f_j(A) = f_j(A)f_i(A)\,[g_i(A)g_j(A)]^{-1}. \tag{8}$$

Carring out suitable multiplications in (8), the commutativity of $r_j(A)$ follows.

Lemma 5 . *If $f(z)$ and $g(z)$ are polynomials, $A \in \mathbb{R}^{n\times n}$, λ is an eigenvalue of A, and $g(A)$ is invertible then $g(\lambda) \neq 0$ and $\frac{f(\lambda)}{g(\lambda)}$ is an eigenvalue of $f(A)g^{-1}(A)$.*

Proof. $g(\lambda)$ is an eigenvalue of $g(A)$. Since $g(A)$ is invertible then $g(\lambda) \neq 0$. There exists $x \in \mathbb{C}^{n\times 1}$, $x \neq 0$ such that the following can be written:

$$\begin{aligned}
Ax &= \lambda x \\
f(A)x &= f(\lambda)x \\
g(A)x &= g(\lambda)x \\
g^{-1}(A)x &= \frac{1}{g(\lambda)}x \\
f(A)g^{-1}(A)x = f(A)\frac{1}{g(\lambda)}x &= \frac{f(\lambda)}{g(\lambda)}x.
\end{aligned}$$

Lemma 6 . *Let $f(z)$ and $g(z)$ be polynomials and $g(A)$ be invertible. If μ is an eigenvalue of $f(A)g^{-1}(A)$ then there exists an eigenvalue λ of A such that $g(\lambda) \neq 0$ and $\mu = \frac{f(\lambda)}{g(\lambda)}$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of A. Then $g(\lambda_i) \neq 0$ $(i = 1, 2, \ldots, n)$ and by Lemma 5, $\frac{f(\lambda_i)}{g(\lambda_i)}$ are eigenvalues of $f(A)g^{-1}(A)$. Therefore, there exists i such that $\mu = \frac{f(\lambda_i)}{g(\lambda_i)}$.

Theorem 7 . *Let $A \in \mathbb{R}^{n\times n}$ and the stability region $\mathbb{D}$ (1) be given. Then the* *following are equivalent :*

i) A *is* $\mathbb{D}$*-stable .*

ii) $g_j(A)$ *are invertible and* $r_j(A) = f_j(A)g_j^{-1}(A)$ *are Hurwitz stable* $(j = 1, 2, \ldots, m)$.

iii) $g_j(A)$ *are invertible and there exists* $P \in \mathbb{R}^{n\times n}$ *,* $P > 0$ *such that*

$$[r_j(A)]^T P + P [r_j(A)] < 0 \, (j = 1, 2, \ldots, m) \, . \tag{9}$$

Proof. The implication iii)$\Longrightarrow$ii) follows from the Lyapunov Theorem.
ii)$\Longrightarrow$i) : Let λ be an arbitrary eigenvalue of A. Then $g_j(\lambda) \neq 0$ and by Lemma 5, $\frac{f_j(\lambda)}{g_j(\lambda)}$ are eigenvalues of $r_j(A)$ $(j = 1, 2, \ldots, m)$. Since $r_j(A)$ are Hurwitz stable, then $Re\frac{f_j(\lambda)}{g_j(\lambda)} < 0$ or $Ref_j(\lambda)\bar{g}_j(\lambda) < 0$ $(j = 1, 2, \ldots, m)$. Thus $\lambda \in \mathbb{D}$.
i)$\Longrightarrow$iii) : Fix arbitrary j. Let μ be an arbitrary eigenvalue of $g_j(A)$.
Let λ_1, $\lambda_2, \ldots, \lambda_n$ be eigenvalues of A. Then $g_j(\lambda_1), \ldots, g_j(\lambda_n)$ are the eigenvalues of $g_j(A)$. Therefore there exists i such that $\mu = g_j(\lambda_i)$. Since A is $\mathbb{D}$-stable $g_j(\lambda_i) = \mu \neq 0$. On the other hand μ is an arbitrary eigenvalue of $g_j(A)$. Consequently $g_j(A)$ is invertible.
By Lemmas 4 and 6 the matrices $r_j(A)$ are Hurwitz and commute $(j = 1, 2, \ldots, m)$. Then by Theorem 2 there exists $P > 0$ such that (9) is true. This completes the proof.

Theorem 7 can be extended to the case where $A \in \mathbb{C}^{n\times n}$ and the polynomials f_j and g_j have complex coefficients. Such an extension is straightforward and is omitted here.

Example 8 *([1]). Let* A *be given as*

$$A = \begin{bmatrix} -94.7 & -47.1 & -41.1 & -2.3 \\ 15.2 & -46.9 & 3.0 & -7.6 \\ 121.0 & 77.9 & 46.3 & 9.1 \\ -116.9 & 65.2 & -54.6 & -4.7 \end{bmatrix}$$

and the region Ω is the shaded region in Fig. 1.

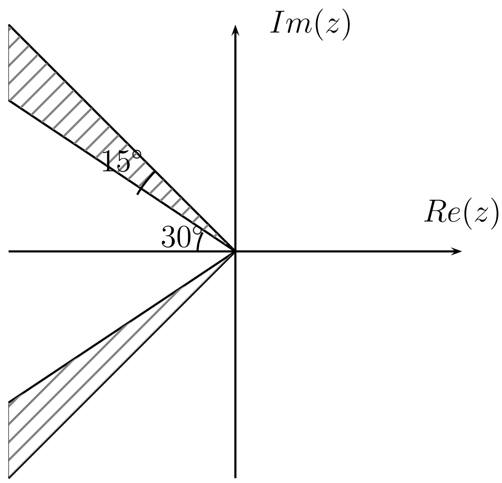

Fig. 1. Sector Ω for Example 8.

This region can be expressed as $\Omega = \{z \in \mathbb{C} : Ref_j(z) < 0, j = 1, 2, 3\}$,
where $f_1(z) = z$, $f_2(z) = -z^2$, $f_3(z) = -z^3$.
The matrices A, $-A^2$, *and* $-A^3$ *are Hurwitz stable. Therefore by Theorem 7 the matrix* A *is* Ω*-stable.*
In [1] this stability is established by finding a common solution $P > 0$ *for (4), which is more difficult problem.*

Example 9 . *Let* A *be given as*

$$A = \begin{bmatrix} 0 & -0.01 & 0.5 \\ 1 & 0 & -1.5 \\ -0.01 & 1 & 1.9 \end{bmatrix}$$

and $\mathbb{D} = \{z \in \mathbb{C} : Ref_j(z).\bar{g}_j(z) < 0, j = 1, 2\}$,

where $f_1(z) = z + 1,\ g_1(z) = z - 1,\ f_2(z) = -z - \frac{1}{2}, g_2(z) = z - \frac{1}{2}$.

The region $\mathbb{D}$ *is the ring* $\left\{(x, y) : \frac{1}{4} < x^2 + y^2 < 1\right\}$. *Here*

$$r_1(A) = \begin{bmatrix} -11.48 & -10.48 & -10.712 \\ 18.409 & 19.616 & 20.8 \\ -20.592 & -20.802 & -20.008 \end{bmatrix}$$

$$r_2(A) = \begin{bmatrix} -8.1846 & -4.6161 & -2.3799 \\ 12.438 & 5.2416 & 2.2451 \\ -8.9358 & -4.4912 & -3.3351 \end{bmatrix}$$

and $r_1(A), r_2(A)$ *are Hurwitz stable. Therefore* A *is* $\mathbb{D}$*-stable.*

3 Stability of a commuting family

In this section for a commuting family we give $\mathbb{D}$-stablity criterion in terms of the existence of a common positive definite solution to a set of Lyapunov inequalities.

The following lemma is taken from [4], [5].

Lemma 10 *([4], [5]). Let $\mathcal{B} \subset \mathbb{C}^{n\times n}$ be a compact set of Hurwitz stable upper triangular matrices. Then there exists $\alpha > 0$ and a positive diagonal matrix D such that*

$$A^* D + DA \leq -\alpha I$$

for all $A \in \mathcal{B}$, where I is the identity matrix.

The following theorem follows from the Schur's triangularization theorem ([6], p.81) and Lemma 10.

Theorem 11 . *Let $\mathcal{F} \subset \mathbb{R}^{n\times n}$ be a compact commuting family. Then $\mathcal{F}$ is Hurwitz stable if and only if there exists $P \in \mathbb{R}^{n\times n}$, $P > 0$ such that for all $F \in \mathcal{F}$*

$$F^T P + PF < 0. \tag{10}$$

It follows from Theorem 7 that for a matrix $A \in \mathbb{R}^{n\times n}$, $\mathbb{D}$-stablity of A is equivalent to the invertibility of all $g_j(A)$ and Hurwitz stability of all $f_j(A)g_j^{-1}(A)$ $(j = 1, 2, \ldots, m)$. Hence we have the following corollary of Theorem 11.

Corollary 12 . *Let $\mathcal{F} \subset \mathbb{R}^{n\times n}$ be a compact commuting family. Then $\mathcal{F}$ is $\mathbb{D}$-stable if and only if for all $F \in \mathcal{F}$ and for all $j = 1, 2, \ldots, m$ the matrix $g_j(A)$ is invertible and there exists $P \in \mathbb{R}^{n\times n}$, $P > 0$ such that for all $F \in \mathcal{F}$, $j = 1, 2, \ldots, m$*

$$\left[f_j\left(F\right) g_j^{-1}(F)\right]^T P + P\left[f_j\left(F\right) g_j^{-1}(F)\right] < 0.$$

4 Common solution for a quadratic polynomial matrix family

In this section for a quadratic polynomial matrix family we give a sufficient condition for the existence of a common solution to the family of Lyapunov inequalities. Let

$$A(t) = A_0 + tA_1 + t^2 A_2, \tag{11}$$

where $t \in [0,1]$, and $A_j \in \mathbb{R}^{n\times n}$ are commuting $(j = 0,1,2)$. It follows from Theorem 11 that the family $\mathcal{A} = \{A(t) : t \in [0,1]\}$ is Hurwitz stable if and only if there exists a common positive definite solution to a set of Lyapunov inequalities (see (10)). Here we give one class of the family (11) for which a common positive definite P exists.

As proved in [2] for Hurwitz stable commutative matrices A and B a common positive definite solution to the Lyapunov inequalities exists and the matrix

$$P = \int_0^\infty \exp(B^T t) \left[\int_0^\infty \exp(A^T \tau) P_0 \exp(A\tau) d\tau \right] \exp(Bt) dt \tag{12}$$

is a common solution, where $P_0 > 0$ is an arbitrary.

Theorem 13 . *Let $A(t)$ (11) be given and A_0, A_1, A_2 be pairwise commutative. Assume that A_0, and $A_0 + A_1 + A_2$ are Hurwitz stable and for the matrix P (12) with $A = A_0$, $B = A_0 + A_1 + A_2$ the matrix inequality*

$$A_2^T P + PA_2 > 0 \tag{13}$$

is true. Then the family $\mathcal{A} = \{A(t) : t \in [0,1]\}$ is Hurwitz stable and P is a common solution to the Lyapunov inequalities for the family $\mathcal{A}$.

Proof. We have to prove the inequality

$$\max_{t\in[0,1]} \lambda_{\max}(A^T(t)P + PA(t)) < 0, \tag{14}$$

where $\lambda_{\max}(.)$ indicates the maximum eigenvalue. We have

$$\begin{aligned}\varphi(t) &\triangleq \lambda_{\max}(A^T(t)P + PA(t)) \\ &= \max_{v\in V} v^T(A^T(t)P + PA(t))v \\ &\triangleq \max_{v\in V} f(t,v),\end{aligned} \tag{15}$$

where $V = \{v \in \mathbb{R}^n : \|v\| = 1\}$, $f(t,v) = v^T\left(A^T(t)P + PA(t)\right)v$. The function $t \to f(t,v)$ is convex. Indeed, by (11), (13)

$$\frac{\partial^2 f}{\partial t^2} = 2v^T(A_2^T P + PA_2)v > 0,$$

so the function $t \to f(t,v)$ is convex and $\varphi(t)$ is also convex (the maximum of a family of convex functions is also convex). Every continuous convex function defined on a closed interval attains its maximum value at the endpoints. Therefore

$$\begin{aligned}\max_{t\in[0,1]} \varphi(t) &= \max\{\varphi(0), \varphi(1)\} \\ &= \max\left\{\lambda_{\max}\left(A(0)^T P + PA(0)\right), \lambda_{\max}\left(A(1)^T P + PA(1)\right)\right\}\end{aligned}$$

We have $A(0) = A_0$, $A(1) = A_0 + A_1 + A_2$. $A(0)$ and $A(1)$ are Hurwitz stable and commutative. Then by Theorem 2 the pair $\{A(0), A(1)\}$ has a common solution to the Lyapunov inequalities and the matrix P (12) with $A = A(0), B = A(1)$ is a common solution, that is $\lambda_{\max}\left(A^T(0)P + PA(0)\right) < 0$, and $\lambda_{\max}\left(A^T(1)P + PA(1)\right) < 0$. Therefore, by (15), $\varphi(t) < 0$ for all $t \in [0,1]$ and (14) is true.

Corollary 14 . *Let* $A(t) = A_0 + tA_1 + t^2 I \quad (t \in [0,1])$ *be given* *where $A_0A_1 = A_1A_0$. Then the family $\{A(t) : t \in [0,1]\}$ is Hurwitz stable and there exists a common $P > 0$ for this family if and only if A_0 and $A_0 + A_1 + I$ are Hurwitz stable.*

Now consider (11), where A_0, A_1, A_2 are not necessarily commutative.
The following theorem can be proved similarly.

Theorem 15 . *Let the family $A(t)$ $(t \in [0,1])$ (11) be given. Assume that A_0 and $A_0+A_1+A_2$ are Hurwitz stable and there exists a common solution $P > 0$ to the Lyapunov inequalities for the pair $\{A_0, A_0 + A_1 + A_2\}$ and assume that this common P satisfies (13). Then the family $\mathcal{A} = \{A(t) : t \in [0,1]\}$is Hurwitz stable and P is a common solution to the Lyapunov inequalities for the family $\mathcal{A}$.*

5 Common solution for two 3×3 dimensional z- matrices

Recall that a real $n \times n$ matrix $A = (a_{ij})$ is said to be z-matrix if $a_{ij} \leq 0$ for all $i \neq j$. The matrix A is called mergelian if $-A$ is z-matrix.
The following properties of z-matrices can be found in [8]:

a) If A is z-matrix and positive stable (i.e. all eigenvalues lie in the open right half plane) then all principle submatrices of A are also positive stable

b) Let A and B be $n \times n$ positive stable z-matrices. Then the segment $[A, B]$ is positive stable if and only if the matrix AB^{-1} has no negative eigenvalues.

Let A and B be two positive stable 2×2 dimensional matrices. Then there exists a positive definite matrix $P \in R^{2\times2}$ such that

$$A^T P + PA > 0 \tag{16}$$
$$B^T P + PB > 0 \tag{17}$$

if and only if the matrices AB and AB^{-1} have no negative eigenvalues [7].

Theorem 16 . *Let A and B be 2×2 dimensional positive stable z-matrices. Then the matrix segment $[A, B]$ is positive stable if and only if there exists a common positive definite solution P of the Lyapunov inequalities (16), (17).*

Proof. $\Longrightarrow$): Since the matrix segment $[A, B]$ is positive stable then by b) the matrix AB^{-1} has no real negative eigenvalues. Now we need to show that the matrix AB has no negative eigenvalues also. Let

$$A = \begin{bmatrix} a & b \\ \\ c & d \end{bmatrix}, B = \begin{bmatrix} m & n \\ \\ p & q \end{bmatrix}, AB = \begin{bmatrix} am + bp & an + bq \\ \\ cm + dp & cn + dq \end{bmatrix}$$

Then $b \leq 0, c \leq 0, n \leq 0, p \leq 0$, and from positive stability of A, B and a) it follows that $a > 0, d > 0, m > 0, q > 0$. AB has the characteristic equation

$$\lambda^2 - \beta\lambda + det(AB) = 0, \tag{18}$$

where $\beta = am + bp + cn + dq > 0$. Hence

$$\lambda = \frac{\beta \pm \sqrt{\beta^2 - 4det(AB)}}{2}.$$

If equation (18) has a negative root then

$$\beta^2 \geq 4det(AB),$$
$$\beta < \sqrt{\beta^2 - 4det(AB)}$$

and from this we get

$$det(AB) < 0. \tag{19}$$

Since $-A$ and $-B$ are Hurwitz stable then $detA > 0$, $detB > 0$ which contradicts (19). Thus the matrix AB has no negative real eigenvalue. Then by (16), (17), it follows that there exists a common positive definite solution of the Lyapunov inequalities.

The implication $\Leftarrow$) is obvious.

We now proceed to the existence problem of a common solution of the Lyapunov inequalities for two positive stable 3×3 dimensional z-matrices.

Let $\widetilde{A} = (a_{ij}), \widetilde{B} = (b_{ij})$ be two positive stable, 3×3 dimensional z-matrices and assume that the matrix segment $[\widetilde{A}, \widetilde{B}]$ is also positive stable (necessary condition for the existence of a common solution and this condition is equivalent to the nonexistence of negative eigenvalues of $\widetilde{A}\widetilde{B}^{-1}$).

Write A and B as

$$\widetilde{A} = \begin{bmatrix} A & & a_{13} \\ & & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \widetilde{B} = \begin{bmatrix} B & & b_{13} \\ & & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

then by a) the matrix segment $[A, B]$ is also positive stable. By Theorem 16 there exists a common solution $P > 0$ of the Lyapunov inequalities for the pair $\{A, B\}$. Define $\widetilde{P} > 0$ as follows:

$$\widetilde{P} = \begin{bmatrix} P & 0 \\ & 0 \\ 0 \;\; 0 & 1 \end{bmatrix}. \tag{20}$$

Theorem 17 . *Let $\widetilde{A}$ and $\widetilde{B}$ be 3×3 dimensional z- matrices and the segment $[\widetilde{A}, \widetilde{B}]$ be positive stable. If $\widetilde{P}$ is defined as in (20) and*

$$\det\left[\widetilde{A}^T\widetilde{P} + \widetilde{P}\widetilde{A}\right] > 0 \tag{21}$$
$$\det\left[\widetilde{B}^T\widetilde{P} + \widetilde{P}\widetilde{B}\right] > 0 \tag{22}$$

then the matrix $\widetilde{P}$ (20) is a common solution of the Lyapunov inequalities for the pair $\{\widetilde{A}, \widetilde{B}\}$.

Proof. Consider $C_1 = \widetilde{A}^T\widetilde{P} + \widetilde{P}\widetilde{A}$, $C_2 = \widetilde{B}^T\widetilde{P} + \widetilde{P}\widetilde{B}$. The matrices C_1 and C_2 are symmetric and have the form

$$C_1 = \begin{bmatrix} A^TP + PA & \star \\ & \star \\ \star & \star\,\star \end{bmatrix}, C_2 = \begin{bmatrix} B^TP + PB & \star \\ & \star \\ \star & \star\,\star \end{bmatrix} \tag{23}$$

Since P is a common solution for the pair $\{A, B\}$ then (16), (17) are satisfied. From this and (21), (22) it follows that $C_1 > 0, C_2 > 0$.

A similar sufficient condition can be formulated using lower principal submatrices of $\widetilde{A}$, and $\widetilde{B}$ but we omit this.

Example 18 . *Consider the positively stable, z-matrices*

$$\widetilde{A}=\begin{bmatrix} 6 & -1 & -1 \\ -2 & 1 & -0.5 \\ -0.5 & -2 & 15 \end{bmatrix}, \widetilde{B}=\begin{bmatrix} 8 & -2 & -1 \\ -2 & 4 & -1 \\ -0.5 & -0.5 & 10 \end{bmatrix}.$$

The matrix segment $\left[\widetilde{A}, \widetilde{B}\right]$ *is positive stable since* $\widetilde{A}\widetilde{B}^{-1}$ *has no negative eigenvalues. Define*

$$A=\begin{bmatrix} 6 & -1 \\ -2 & 1 \end{bmatrix}, B=\begin{bmatrix} 8 & -2 \\ -2 & 4 \end{bmatrix}, P=\begin{bmatrix} 12.6762 & -2.7909 \\ -2.7909 & 5.6962 \end{bmatrix}$$

where $P > 0$ *is the common solution for the pair* $\{A, B\}$*. For the matrix*

$$\widetilde{P}=\begin{bmatrix} 12.6762 & -2.7909 & 0 \\ -2.7909 & 5.6962 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(21), (22) are satisfied and by Theorem 17 the matrix $\widetilde{P}$ *is a common solution for the pair* $\{\widetilde{A}, \widetilde{B}\}$*.*

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