Semigroup of matrices acting on the max-plus projective space

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Abstract

We investigate the action of semigroups of $d \times d$ matrices with entries in the max-plus semifield on the max-plus projective space. Recall that semigroups generated by one element with projectively bounded image are projectively finite and thus contain idempotent elements.

In terms of orbits, our main result states that the image of a minimal orbit by an idempotent element of the semigroup with minimal rank has at most $d!$ elements. Moreover, each idempotent element with minimal rank maps at least one orbit onto a singleton.

This allows us to deduce the central limit theorem for a stochastic recurrent sequences driven by independent random matrices that take countably many values, as soon as the semigroup generated by the values contains an element with projectively bounded image.

1 Introduction

1.1 Definitions

In this article, we investigate the action of semigroups of $d \times d$ matrices with entries in the max-plus semifield. This semifield will be denoted by $\mathbb{R}_{\max}$ and is the set $\mathbb{R} \cup \{-\infty\}$ equipped with operations $\boxplus = \max$ and $\boxtimes = +$. The set of all square matrices with size d will be $\mathbb{R}_{\max}^{d\times d}$.

Those matrices have been extensively studied since the sixties. An early reference is [CG79]. For a recent introduction, see [HOvdW06]. Products of matrices or vectors with appropriate size are given by the following formula

$$(A \boxtimes B)_{ij} = \boxplus_k A_{ik} \boxtimes B_{kj} = \max_k (A_{ik} + B_{kj}). \tag{1}$$

As in the usual linear algebra, one can identify the matrix A and the function from $\mathbb{R}_{\max}^d$ to itself that maps x on $A \boxtimes x$.

A matrix $A \in \mathbb{R}_{\max}^{d\times d}$ is called *regular* if it has at least one finite entry in each row. In formula $\forall i, \exists j, A_{ij} \neq -\infty$. Regular matrices are exactly those that map

*This work was supported by the ANR project MASED (06-JCJC-0069).

$\mathbb{R}^d$ into itself. In the sequel, we will only consider regular matrices, and identify them with the map they define on $\mathbb{R}^d$, which are known to be non-expanding with respect to the infinity norm ([CT80]).

For any $a \in \mathbb{R}^d$, the max-plus line $\mathbb{R}_{\max} \boxtimes a$ is the affine line that goes through a and is parallel to the unit vector $\mathbf{1} = (1, \cdots, 1)'$ augmented by $(-\infty)^d$. Therefore we will call *max-plus projective space* $\mathbb{PR}^d_{\max}$ the quotient space of $\mathbb{R}^d$ [1] by the equivalence relation $\sim$ defined by $x \sim y$ if $x - y$ is proportional to $\mathbf{1}$. Moreover, $\overline{x}$ will be the equivalence class of x.

The mapping $\overline{x} \mapsto (x_i - x_j)_{i<j}$ embeds $\mathbb{PR}^d_{\max}$ onto a subspace of $\mathbb{R}^{\frac{d(d-1)}{2}}$ with dimension $d-1$. The infinity norm of $\mathbb{R}^{\frac{d(d-1)}{2}}$ therefore induces a distance on $\mathbb{PR}^d_{\max}$ which will be denoted by δ. A direct computation shows that $\delta(\overline{x}, \overline{y}) = \max_i(x_i - y_i) + \max_i(y_i - x_i)$. By a slight abuse, we will also write $\delta(x, y)$ for $\delta(\overline{x}, \overline{y})$.

Regular matrices define maps from $\mathbb{PR}^d_{\max}$ to itself. Such maps are called *projective maps* and are non-expanding with respect to δ (Mairesse [Mai97]). The projective map defined by A will be denoted by $\overline{A}$. We are interested in the action of semigroups of projective maps on $\mathbb{PR}^d_{\max}$.

1.2 Motivations

Our primary motivation to study the orbits in the projective space is the understanding of the so-called stochastic max-plus linear systems. These are the systems, whose state space is $\mathbb{R}^d$ and the state $x(n+1)$ of the system at time $n+1$ follows from the state at time n by the recurrence relation

$$x(n+1) = A_n \boxtimes x(n). \tag{2}$$

Those systems appear in the modeling of a wide class of discrete event systems, such as transportation systems (e.g. [HdV01]), computer networks (e.g. [BH00]) or production lines (e.g. [CDQV85]). For the sake of simplicity, we will restrict our attention to the case in which $(A_n)_{n \in \mathbb{N}}$ is a sequence of independent identically distributed (i.i.d. for short) random regular matrices that take countably many values. In formulas, we assume that there is a countable set V of regular $d \times d$ matrices such that $\mathbb{P}(A_n \in V) = 1$ for all n, and for any integers $n_1 < n_2 \cdots < n_k$ and matrices $B_i \in V$, we have $\mathbb{P}(\forall i, A_{n_i} = B_i) = \Pi_{i=1}^k \mathbb{P}(A_1 = B_i)$.

In the deterministic case (i.e. $A_n = A$ for all n), those system are well described. Indeed, when A is *projectively bounded* (meaning that the image of $\overline{A}$ is bounded), the semigroup generated by $\overline{A}$ is finite ([CDQV83, CDQV85]), which implies that $(x(n))_{n \in \mathbb{N}}$ is ultimately pseudo-periodic. In formulas, there are a real ρ, and to integers c and N such that $x(n+c) = \rho^{\boxtimes c} \boxtimes x(n)$ for any $n \geq N$.

[1] We defined the projective space as the quotient of $\mathbb{R}^d$ and not of $\mathbb{R}^d_{\max} \backslash \{-\infty^d\}$, so that the projective space is not compact. Another choice would have been to work with the isometric field $(\mathbb{R}^+, \max, \times)$ and the usual projective space, restrited to its nonnegative quadrant. Since we will see in Remark 1.1 that we must restrict to bounded sets, it is more convenient to work with our definition.

In the stochastic case, Mairesse introduced the notion of *memory loss property (MLP)*, which means that there is an integer m such that the matrix $A_m \boxtimes \cdots \boxtimes A_0$ has *rank one* (i.e. the map it defines on $\mathbb{PR}^d_{\max}$ is a constant) with positive probability. This property implies a variety of stability theorems (see [Mai97, GH00, Mer05, Mer07]) for $(x(n))_{n\in\mathbb{N}}$.

When A_0 takes countably many values, $(A_n)_{n\in\mathbb{N}}$ has MLP if and only if the semigroup generated by those values contains an element with rank one. This implies that there is only one minimal orbit in $\mathbb{PR}^d_{\max}$ under the action of this semigroup. In [Mer04], we have shown that the semigroup generated by two finite matrices A and B has a rank 1-element, except if the pair (A, B) is an element of a finite union of hyperplanes of $(\mathbb{R}^{d\times d})^2$.

This result says that MLP is generic in a strong sense among sequences of matrices that takes countably many values. Moreover, we have a similar result for arbitrary sequences. But what about the degenerate case ? This question is interesting not only theoretically, but also from an applied point of view. Indeed, as dimension d becomes large, the number of conditions to check to prove the MLP grows quicker than $d!$. Moreover, those conditions depend on the values of the matrices, that are not always precisely known. Therefore, we looked for a simpler condition that would only depend on the place of finite entries in the matrices.

A natural candidate for this condition is the existence of an integer m such that $A_m \boxtimes \cdots \boxtimes A_0$ is projectively bounded with positive probability. Since a matrix is projectively bounded if and only if the entries of each of its column vectors are either all finite or all equal to $-\infty$, this condition only depends on the place of finite entries in the matrices. It is a natural condition for several reasons. First, it ensures that the limit of $\left(\frac{1}{n}x_i(n)\right)_{n\in\mathbb{N}}$ exists, is deterministic, and is the same for all i (see [Hon01]). Second, it is a translation into the max-plus case of the hypothesis that ensures the Central Limit Theorem (CLT) for usual product of nonnegative matrices (see [Hen97]). Note that the proof of the CLT can not be adapted with this hypothesis, because it relies on the fact that projectively bounded matrices are strictly contracting with respect to Hilbert distance. The condition that $A_m \boxtimes \cdots \boxtimes A_0$ is strictly contracting with respect to δ is exactly the MLP, which also ensure CLT (see [Mer05, Mer07]).

To deal with the projective boundedness condition, we introduce the notion of *pseudo-primitive* semigroups of projective maps, that is semigroups that contain at least one projectively bounded element[2]. The main theorem of this paper, to be stated in the next section, gives an insight into the orbit of such semigroups, which will allow us to deduce the CLT for $(x(n))_{n\in\mathbb{N}}$.

1.3 Statements

Our main result is the following

Theorem 1.1. *Let $\mathcal{S}$ be a pseudo-primitive semigroup of max-plus projective maps and P be a bounded element of $\mathcal{S}$ such that $P \circ P = P$ and such that*

[2] In [Gau94], Gaubert calls primitive the finitely generated semigroups all elements but finitely many have only finite entries

$dim(\mathrm{Im}P)$ is minimal in $\mathcal{S}$. Then all elements in $P\mathcal{S}$ have a common fixed point.

Moreover $\cap_{A\in\mathcal{S}}\mathrm{Im}PA$ is a nonempty compact set on which $P\mathcal{S}$ acts as a finite group of isometries, with at most $(dim\mathrm{Im}P)!$ elements.

In terms of orbits, it says that the image of a minimal orbit by an idempotent element of the semigroup with minimal tropical rank[3] has at most $(dim\mathrm{Im}P)!$ elements. Moreover, each idempotent element with minimal rank maps at least one orbit onto a singleton.

Remarks 1.1.

1. The dimension of the image of a max-plus linear map is well defined, since such a map is affine on convex sets with nonempty interior whose union is the whole initial set. By a slight abuse of notation, we will write $dim\mathrm{Im}\overline{A}$ for $dim\mathrm{Im}A$ it is well defined, because if $\overline{A}=\overline{B}$, then $\mathrm{Im}A=\mathrm{Im}B$.

2. It is necessary to assume that the semigroup is pseudo-primitive. Indeed, consider the semigroup of all regular diagonal matrices $\{A\in\mathbb{R}_{\max}^{d\times d}|A_{ij}\neq-\infty\Leftrightarrow i=j\}$ and $\mathcal{S}$ the semigroup of projective maps it defines. It is actually the group of all translations of $\mathbb{PR}_{\max}^d$, thus $P\mathcal{S}=\mathcal{S}$ for all $P\in\mathcal{S}$. On the other hand all its elements but the identity have no fixed point.

To state the corollary, let us recall that the top Lyapunov exponent of an i.i.d. sequence of random regular matrices is the limit of the sequence $\left(\frac{1}{n}\max_{i,j}\left(A_n\boxtimes\cdots\boxtimes A_0\right)_{ij}\right)_{n\in\mathbb{N}}$, which converges almost surely and in mean, as soon as $A_1\boxtimes 0$ is integrable.

Corollary 1.2 (CLT)**.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed random regular matrices that take countably many values and γ be its top Lyapunov exponent.*

If the semigroup generated by those values is pseudo-primitive, then for any initial vector X_0 the sequence $\left(\frac{1}{\sqrt{n}}\left(A_n\boxtimes\cdots\boxtimes A_0\boxtimes X_0-n\gamma\boxtimes\mathbf{1}\right)\right)_{n\in\mathbb{N}}$ converges in law to a normal distribution with dimension 1.

Remarks 1.2.

1. This result proves the CLT only for matrices A_n that take countably many values. On the other hand, the MLP can be stated for any matrices and implies CLT. (See [Mer07, Mer05]) We therefore expect that CLT holds as soon as $A_m\boxtimes\cdots\boxtimes A_0$ is projectively bounded for some m. Unfortunately, the proof of Corollary 1.2 relies on the existence of a given matrix P that should appear as product $A_m\boxtimes\cdots\boxtimes A_0$ with positive probability. Thus, this proof can not be extended right away to arbitrary matrices A_n.

2. Theorem 1.1 could be used to prove other limit theorems than the CLT of Corollary 1.2, such as Local limit theorem, renewal theorem, or CLT with rate. It also works for sequences of dependant matrices, that satisfy suitable mixing hypotheses. Most of the needed estimates are availabe in [Mer05, Mer07].

[3]For a survey about the several notions of max-plus ranks, see [DSS05]

The remaining of this article is devoted to the proof of Theorem 1.1 and its corollary. In the next section, we recall some elements of the asymptotic theory of matrices in $\mathbb{R}_{\max}^{d\times d}$. In Section 3, we prove the theorem in a nice but rare case, where all matrices have maximal rank. In section 4, we deduce the general case from this nice case. Finally, Section 5 is devoted to the proof of the corollary.

2 Asymptotic theory of matrices

In this section, we briefly review some elements of spectral and asymptotic theory of max-plus matrices. For a complete exposition, see Baccelli et al. [BCOQ92] or Heidergott et al. [HOvdW06].

Definition 2.1. A circuit on a directed graph is a closed path on the graph. Let A be a square matrix of size d with entries in $\mathbb{R}_{\max}$.

i) The *graph* of A is the directed weighted graph whose nodes are the integers between 1 and d and whose arcs are the (i,j) such that $A_{ij} > -\infty$[4]. The graph of A will be denoted by $\mathcal{G}(A)$ and the set of its elementary circuits by $\mathcal{C}(A)$.

ii) The *weight* of the path $pth = (i_1, \cdots, i_n, i_{n+1})$ is $w(A, pth) := \sum_{j=1}^{n} A_{i_j i_{j+1}}$. Its length is $|pth| := n$. Its *average weight* is $aw(A, pth) := \frac{w(A,pth)}{|pth|}$.

iii) The max-plus *spectral radius* of A is $\rho_{\max}(A) := \max_{c\in\mathcal{C}(A)} aw(A, c)$.

iv) The *critical graph* of A is obtained from $\mathcal{G}(A)$ by keeping only the nodes and arcs which belong to circuits with average weight $\rho_{\max}(A)$. It will be denoted by $\mathcal{G}^c(A)$.

v) The *cyclicity* of a strongly connected graph is the greatest common divisor of the length of its circuits. The cyclicity of a general graph is the least common multiple of the cyclicities of its strongly connected components. The cyclicity of A is the cyclicity of $\mathcal{G}^c(A)$ and is denoted by $c(A)$.

We will need some results from spectral theory. If $\lambda \in \mathbb{R}_{\max}$ and $V \in \mathbb{R}_{\max}^{d} \backslash \{(-\infty)^d\}$ satisfy the equation $A \boxtimes V = \lambda \boxtimes V$, we say that λ is an *eigenvalue* of A and V is an *eigenvector*.

For every $A \in \mathbb{R}_{\max}^{d\times d}$, the matrix $\tilde{A}$ defined by $\tilde{A}_{ij} = A_{ij} - \rho_{\max}(A)$ satisfies $\rho_{\max}(\tilde{A}) = 0$ and $A = \rho_{\max}(A) \boxtimes \tilde{A}$. In the sequel, we will therefore only deal with the case $\rho_{\max}(A) = 0$.

For every $A \in \mathbb{R}_{\max}^{d\times d}$ with $\rho_{\max}(A) \le 0$, we set:

$$A^{+} := \boxplus_{n\ge 1} A^{\boxtimes n}.$$

Proposition 2.2. *Let A be a projectively bounded matrix in $\mathbb{R}_{\max}^{d\times d}$. We have the following.*

[4] Some previous author used the isomorphic graph with weight A_{ji} on arc (i,j) but this definition has proved to be more convenient in random matrices, see [Mer08]

i) The spectral radius $\rho_{\max}(A)$ is the only eigenvalue of A.

ii) If $\rho_{\max}(A) = 0$ and E is a set of integers that contains exactly one vertex in each strongly connected component (s.c.c.) of $\mathcal{G}^c(A)$, then $\{A^+_{\cdot i} : i \in E\}$ is a minimal generating set of the eigenspace of A.

iii) If $\rho_{\max}(A) = 0$ and $c(A) = 1$, then there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $A^{\boxtimes n} = Q$.

Corollary 2.3. *If A is a projectively bounded map, then there is an $n \in \mathbb{N}$ such that $A^{\boxtimes n} \boxtimes A^{\boxtimes n} = \rho_{\max}(A^{\boxtimes n}) \boxtimes A^{\boxtimes n}$.*

Proof. Those results are due to Cuninghame-Green [CG79] (for i)) and to Cohen et al. [CDQV83, CDQV85] when $\mathcal{G}(A)$ is strongly connected. As we already noticed, the entries of column vectors of a projectively bounded matrix are either all finite or all equal to $-\infty$. Therefore, up to renumbering the coordinates, A is of the form $A = \begin{pmatrix} B & -\infty \\ C & -\infty \end{pmatrix}$, with finite matrices B and C.

Therefore, we have

$$A \boxtimes \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B \boxtimes X_1 \\ C \boxtimes X_1 \end{pmatrix} \text{ and } A^{\boxtimes n} = \begin{pmatrix} B^{\boxtimes n} & -\infty \\ C \boxtimes B^{\boxtimes (n-1)} & -\infty \end{pmatrix}$$

and the results hold for projectively bounded A too. □

3 Nice semigroups

3.1 Statement

Finite matrices $A \in \mathbb{R}^{d \times d}$ such that $\dim(\mathrm{Im} A) = d$ are called *strongly regular* (**s.r.**) by Cuninghame-Green. A semigroup of strongly regular matrices is called *nice*.

This section is devoted to the proof of the following theorem, which implies Theorem 1.1 for semigroups of projective maps defined by a nice semigroup of matrices.

Theorem 3.1. *If $\mathcal{S} \subset \mathbb{R}^{d \times d}_{\max}$ is a nice semigroup of matrices, then all the elements of $\mathcal{S}$ have a common eigenvector.*

Moreover $\cap_{A \in \mathcal{S}} \mathrm{Im} A$ is a nonempty convex projectively compact set on which $\mathcal{S}$ is a finite group of affine isometries embedded in the permutation group S_d.

To prove this statement, we first recall or adapt a few results about strongly regular matrices. This will be the subject of the next subsection. In the following subsection, we show that $\cap_{A \in \mathcal{S}} \mathrm{Im}(A)$ is a projectively compact convex invariant set, on which the matrices acts as affine isometries. Finally we conclude the proof of theorem 3.1.

3.2 Strongly regular matrices

To study strongly regular matrices, we will consider max-plus matrices as piecewise affine maps. To any mapping τ from $\{1, \cdots d\}$ to itself, let us associate the set $E_\tau(A) := \{x | \forall i, \forall j \neq \tau(i), A_{ij} + x_j < A_{i\tau(i)} + x_{\tau(i)}\}$ and the affine map A_τ defined by $(A_\tau x)_i := A_{i\tau(i)} + x_{\tau(i)}$.

For every permutation σ of $\{1, \cdots d\}$, let us set $w(A, \sigma) := \sum_i A_{i\sigma(i)}$ so that the max-plus permanent of matrix A reads $Perm(A) := \max_{\sigma \in \mathrm{S}_d} w(A, \sigma)$.

The following proposition is obvious:

Proposition 3.2.

1. *For any τ, $E_\tau(A)$ is an open convex set and A is equal to A_τ on its closure.*
2. *$\mathbb{R}^d$ is the union of the closures of the $E_\tau(A)$.*
3. *$A \in \mathbb{R}^{d\times d}$ is **s.r.** if and only if there is a $\tau \in S_d$, such that $E_\tau(A) \neq \emptyset$.*

If $E_\tau(A) \neq \emptyset$, then take an element x in $E_\tau(A)$, a permutation σ and sum the inequality $A_{i\sigma(i)} + x_{\sigma(i)} \leq A_{i\tau(i)} + x_{\tau(i)}$ over i. It proves that $w(A, \tau) \geq w(A, \sigma)$. Moreover, if $\sigma \neq \tau$, then one of the inequalities is strict. Thus τ is the unique $\sigma \in \mathrm{S}_d$ that maximizes $w(A, \sigma)$. Let us denote this *permutation* τ_A and let $S(A)$ *be* $A(E_{\tau_A}(A))$.

Conversely, Butkovic proves in [But00, Theorems 3.3 and 4.1] that if the permutation $\sigma \in \mathrm{S}_d$ that maximizes $w(A, \sigma)$ is unique, then $dim(\mathrm{Im}(A)) = d$. This has the following consequence that will be important for our proof.

Lemma 3.3. *Let P be a projectively bounded matrix with $P \boxtimes P = P$. Then* $\dim(\mathrm{Im}P)$ *is the number of s.c.c in $\mathcal{G}^c(P)$.*

Proof. Let P be a projectively bounded matrix with $P \boxtimes P = P$ and r be the number of s.c.c in $\mathcal{G}^c(P)$. Since $P \boxtimes P = P$, we deduce from Proposition 2.2 i) that $\rho_{\max}(P) = 0$, so that $\mathrm{Im}(P)$ is the eigenspace of P and $dim(\mathrm{Im}P) \leq r$, because of Proposition 2.2ii) .

Conversely, consider a subset E of $\{1, \cdots d\}$ with exactly one element in each s.c.c of $\mathcal{G}^c(P)$ and the submatrix $\hat{P}$ of P whose indices are in E. It has zeros on its diagonal, because $P = P^+$ and $P^+_{ii} = 0$ whenever i is in $\mathcal{G}^c(P)$. This says that $\mathcal{G}^c(\hat{P})$ has a loop on each node. On the other hand, there is no other circuit in $\mathcal{G}^c(\hat{P})$, because otherwise its nodes would be in the same s.c.c. of $\mathcal{G}^c(P)$. Therefore, the identity is the only permutation of E with maximal weight relatively to $\hat{P}$. Thus $\hat{P}$ is strongly regular, and its image has dimension r. Therefore, there is an open subset U of $\mathbb{R}^E$, such that $\hat{P}U$ has dimension r.

Without loss of generality, we assume that $E = \{1, \cdots r\}$. For $x = (x_1, \cdots, x_d)$, we denote $(x_1, \cdots, x_r)$ by $\hat{x}$. If the x_i for $i > r$ are small enough, then we have $\widehat{P \boxtimes x} = \hat{P} \boxtimes \hat{x}$, thus there is a positive number M, such that $P(U \times]-2M, -M[)$ has dimension greater than r. This proves that $dim(\mathrm{Im}P) \geq r$ and concludes the proof. $\square$

Butkovic also shows that $S(A)$ is the so-called simple image set of A, i.e. the set of all vectors that have a unique preimage by A. The following proposition sums up a few basic results on strongly regular matrices that are implicitly in [But00] but follow easily from our definition.

Proposition 3.4. *If A and B are two finite matrices such that $A \boxtimes B$ is **s.r.**, then so are A and B, and we have:*

1. $(A \boxtimes B)_{\tau_{A \boxtimes B}} = A_{\tau_A} \circ B_{\tau_B}$
2. $\tau_{A \boxtimes B} = \tau_A \circ \tau_B$
3. $Perm(A \boxtimes B) = Perm(A) \boxtimes Perm(B)$
4. $S(A \boxtimes B) = S(A) \cap A_{\tau_A} S(B)$

Proof. Since both A and B are piecewise affine, $(A \boxtimes B)_{\tau_{A \boxtimes B}}$ is a compostion of two such maps, say A_{τ_1} and B_{τ_2}. But $(A \boxtimes B)_{\tau_{A \boxtimes B}}$ has rank d, thus so have A_{τ_1} and B_{τ_2}. Therefore A and B are **s.r.** $\tau_1 = \tau_A$ and $\tau_2 = \tau_B$, which proves 1. Moreover, we have $E_{\tau_{A \boxtimes B}}(A \boxtimes B) = \{x \in E_{\tau_B}(B) | B_{\tau_B}(x) \in E_{\tau_A}(A)\}$. From this relation, we deduce 4.

For any **s.r.** C and any index i, $\tau_C(i)$ is the index of the only coordinate of $C_{\tau_C}(x)$ that depends on x_i. Applying this result to B, A and $A \boxtimes B$, we deduce 2 from 1.

For any **s.r.** matrix C and any $x \in \mathbb{R}^d$, the permanent satisfy $Perm(C) = \sum_i (C_{\tau_C}(x))_i - \sum_i x_i$. Applying this result to B, A and $A \boxtimes B$, we deduce 3 from 1. □

Corollary 3.5. *If all powers of A are **s.r.**, then $\rho_{\max}(A) = \frac{1}{d} Perm(A)$*

Proof. Let n be an integer given by Corollary 2.3. Then, we have $A^{\boxtimes 2n} = \rho_{\max}(A)^{\boxtimes n} \boxtimes A^{\boxtimes n}$, thus we have

$$Perm(A^{\boxtimes 2n}) = d\rho_{\max}(A^{\boxtimes n}) \boxtimes Perm(A^{\boxtimes n}). \tag{3}$$

But, because of the proposition, $Perm(A^{\boxtimes 2n}) = 2nPerm(A)$ and $Perm(A^{\boxtimes n}) = nPerm(A)$, so that Equation (3) becomes $nPerm(A) = d\rho_{\max}(A^{\boxtimes n})$. Since $\rho_{\max}(A^{\boxtimes n}) = n\rho_{\max}(A)$, this concludes the proof. □

Because of Proposition 3.4, it has the following consequence.

Corollary 3.6. *If A and B are two finite matrices in a nice semigroup, then $\rho_{\max}(A \boxtimes B) = \rho_{\max}(A) \boxtimes \rho_{\max}(B)$.*

The following result will be crucial for our proof.

Theorem 3.7 (Butkovic [But00])**.** *If P is **s.r.** and $P^{\boxtimes 2} = P$, then* $\mathrm{Im} P = cl(S(P)) = Fix(P)$.

Proof. In [But00, Theorem 4.2], the same result is stated for **s.r.** matrices A with only zeros on the diagonal and such that all circuit of $\mathcal{G}(A)$ with length greater or equal two have negative weight. These matrices are exactly those with spectral radius equal to 0 and whose critical graph has d s.c.c.

But according to Lemma 3.3, this number of s.c.c. is the dimension of $\mathrm{Im}A$ that is d, because P is **s.r.**. Since $P^{\boxtimes 2} = P$ implies $\rho_{\max}(P) = 0$, the hypotheses of this Lemma implies those of [But00, Theorem 4.2], which concludes the proof. $\square$

Remark 3.1. The Kleene star of a matrix A is defined as $A^* := \bigoplus_{n\in\mathbb{N}} A^{\boxtimes n}$. If $P^{\boxtimes 2} = P$, then $P^* = P^{\boxtimes 0} \boxtimes P$ In this proof, we noticed that **s.r.** matrices P such that $P^{\boxtimes 2} = P$, have zeros on the diagonal. There are therefore equal to their Kleene star. The importance of such matrices has been emphasized in [Ser08].

In the previous proof, we noticed that $\mathcal{G}^c(P)$ contains all nodes of $\mathcal{G}(A)$ and that $\rho_{\max}(P) = 0$. Therefore $w(P, Id) = 0$. But, according to Corollary 3.5 $Perm(P) = d\rho_{\max}(P) = 0$ thus, $\tau_P = Id$. Now, $P_{\tau_P} = P_{Id}$ which falls down to $P_{\tau_P} = Id$ once we noticed that all diagonal elements of P are equal to $\rho_{\max}(P)$, that is to 0.

This gives the following lemma, which will be used extensively in the next subsection.

Lemma 3.8. *If $P \in \mathbb{R}_{\max}^{d\times d}$ is **s.r.** and satisfy $P^{\boxtimes 2} = P$, then τ_P is the identity on $\{1, \cdots d\}$, and A_{τ_P} is the identity on $\mathbb{R}^d$*

3.3 Proof of the nice case

In this section, we conclude the proof of Theorem 3.1. This proof is split into two lemmas, each of which corresponds to a statement of the theorem.

To each $d \times d$ matrix A we associate the normalized matrix $\tilde{A}$, defined by $\tilde{A}_{ij} = A_{ij} - \rho_{\max}(A)$. Since $A = \rho_{\max}(A) \boxtimes \tilde{A}$, it defines the same projective map and has the same image as A. Since $\sum_i \tilde{A}_{i\tau_{\tilde{A}}(i)} = d\rho_{\max}(\tilde{A}) = 0$, the hyperplane $\Sigma := \{x \in \mathbb{R}^d | \sum_i x_i = 0\}$ is closed under the action of the $\tilde{A}_{\tau_{\tilde{A}}}$.

To a semigroup $\mathcal{S}$, we associate $\tilde{\mathcal{S}} := \left\{\tilde{A} | A \in \mathcal{S}\right\}$. Because of Corollary 3.6, if $\mathcal{S}$ is nice, then $\tilde{\mathcal{S}}$ is also a nice semigroup. Because of Proposition 3.4, so is $\{A_{\tau_A} : A \in \tilde{\mathcal{S}}\}$. It is even a group, as the next lemma states.

Lemma 3.9. *If $\mathcal{S}$ is a nice semigroup, then $\tilde{\mathcal{S}}$ is also a nice semigroup and $\{A_{\tau_A} | A \in \tilde{\mathcal{S}}\}$ is a group of affine isometries that preserves Σ.*

Proof. The only thing to prove is that the inverse of A_{τ_A} is in $\{A_{\tau_A} | A \in \tilde{\mathcal{S}}\}$.

To see this, apply Corollary 2.3 to $\overline{A}$. This gives an n such that $\overline{A^{\boxtimes 2n}} = \overline{A^{\boxtimes n}}$. Since $A \in \tilde{\mathcal{S}}$, $\overline{A^{\boxtimes n}} \in \tilde{\mathcal{S}}$ and the last equation becomes $A^{\boxtimes 2n} = A^{\boxtimes n}$. Now, Lemma 3.8 states that $A^{\boxtimes n}_{\tau_{A^{\boxtimes n}}} = Id$, so that $A^{\boxtimes n-1}_{\tau_{A^{\boxtimes n-1}}} = A^{-1}_{\tau_A}$. $\square$

For any $F \subset \mathbb{R}^d$ and $\epsilon > 0$ we denote by F^ϵ the ϵ-neighborhood of F. In formula:

$$F^\epsilon := \{x \in \mathbb{R}^d | \exists y \in F \|x - y\|_\infty < \epsilon\}.$$

Lemma 3.10. *Let $\mathcal{S}$ be a **nice** semigroup. nd set $I := \cap_{A\in\mathcal{S}}\mathrm{Im}A$ and $\Sigma = \{x \in \mathbb{R}^d | \sum_i x_i = 0\}$. Then the following assumptions hold.*

1. *For any $P_1, P_2 \in \mathcal{S}$ such that $P_i^{\boxtimes 2} = P_i$, there is an n such that $Q = (P_1 \boxtimes P_2)^{\boxtimes n}$ satisfy $Q^{\boxtimes 2} = Q$ and $\mathrm{Im}P_1 \cap \mathrm{Im}P_2 = \mathrm{Im}Q$.*
2. *$I = \cap_{P\in\tilde{\mathcal{S}}, P\boxtimes P = P}\mathrm{Im}P = \cap_{A\in\mathcal{S}} cl(S(A))$.*
3. *The intersection $K := \cap_{A\in\mathcal{S}}\mathrm{Im}A \cap \Sigma$ is a nonempty compact convex set.*
4. *For any $\epsilon > 0$, there is a matrix $C \in \mathcal{S}$ such that $\mathrm{Im}C \subset I^\epsilon$.*

Proof.

1. Let n be the integer given by Corollary 2.3 applied to the projective map defined by $P_1 \boxtimes P_2$. Because of Corollary 3.6, $\rho_{\max}((P_1\boxtimes P_2)^{\boxtimes n}) = 0$, so that $(P_1\boxtimes P_2)^{\boxtimes 2n} = (P_1\boxtimes P_2)^{\boxtimes n}$. Now, apply recursively Proposition 3.4 4., taking into account lemma 3.8. This says that $S((P_1\boxtimes P_2)^{\boxtimes n}) = S(P_1) \cap S(P_2)$. Because of Theorem 3.7, it leads to $\mathrm{Im}((P_1\boxtimes P_2)^{\boxtimes n}) \subset \mathrm{Im}P_1 \cap \mathrm{Im}P_2$.

   On the other hand, each P_i acts as the identity on $\mathrm{Im}P_i$, thus both P_i act as the identity on $\mathrm{Im}P_1 \cap \mathrm{Im}P_2$ and therefore $\mathrm{Im}P_1 \cap \mathrm{Im}P_2 \subset \mathrm{Im}((P_1\boxtimes P_2)^{\boxtimes n})$.

2. First, let us notice that $I = \cap_{A\in\tilde{\mathcal{S}}}\mathrm{Im}A$. Because of Corollary 2.3,

   $$I = \cap_{P\in\tilde{\mathcal{S}}, P\boxtimes P = P}\mathrm{Im}P.$$

   But for any $P \in \tilde{\mathcal{S}}$ such that $P\boxtimes P = P$, $\mathrm{Im}P = cl(S(P))$. Since $S(A) = S(\tilde{A})$ and $S(A^{\boxtimes n}) \subset S(A)$, Corollary 2.3 concludes the proof of this item.

3. For any $P \in \tilde{\mathcal{S}}$ such that $P\boxtimes P = P$, $\mathrm{Im}P \cap \Sigma = cl(S(P)) \cap \Sigma$ is a nonempty compact convex set. Because of item 1, the intersection of finitely many projective images of such $\mathrm{Im}P \cap \Sigma$ is nonempty, thus their overall intersection is also nonempty. It is convex as the intersection of convex sets and compact as the intersection of compact sets.

4. Take any $P \in \tilde{\mathcal{S}}$, such that $P\boxtimes P = P$. To any $x \in \Sigma\backslash K$, we associate an open neighborhood U_x as follows. According to item 2, there exists a $P_x \in \tilde{\mathcal{S}}$ such that $P_x\boxtimes P_x = P_x$ and $x \notin \mathrm{Im}P_x$ and we set $U_x = \mathbb{R}^d\backslash\mathrm{Im}P_x$.

   Now the compact set $\mathrm{Im}P \cap \Sigma$ is covered by K^ϵ and the U_x, so that there a subcovering by K^ϵ and say $U_{x_1} \cdots U_{x_n}$. In formula:

   $$\mathrm{Im}P \cap \Sigma \subset K^\epsilon \cup \bigcup_{i=1}^{n} (\mathbb{R}^d\backslash\mathrm{Im}P_{x_i}).$$

   Applying n times the first item of this lemma, we get a matrix $Q \in \tilde{\mathcal{S}}$, such that $\mathrm{Im}Q = \mathrm{Im}P \cap \bigcap_{i=1}^n \mathrm{Im}P_{x_i}$. Now, take $C \in \mathcal{S}$ such that $\tilde{C} = Q$. By construction, we have $\mathrm{Im}C \cap \Sigma = \mathrm{Im}Q \cap \Sigma \subset K^\epsilon$, and thus $\mathrm{Im}C \subset I^\epsilon$

□

Lemma 3.11. *Let $\mathcal{S}$ be a* ***nice*** *semigroup of matrices and set $I := \cap_{A\in\mathcal{S}}\mathrm{Im}A$ and $\Sigma = \{x \in \mathbb{R}^d | \sum_i x_i = 0\}$. Then, we have the following.*

1. *Any $A \in \mathcal{S}$ coincides with A_{τ_A} on I.*
2. *I is closed under the action of every $A \in \mathcal{S}$ and $I \cap \Sigma$ is closed under the action of every $A \in \tilde{\mathcal{S}}$*
3. *All $A \in \mathcal{S}$ have a common eigenvector in I.*
4. *The mapping $A \mapsto \tau_A$ embeds the restrictions of the element of $\tilde{\mathcal{S}}$ to I into the permutation group S_d.*

Proof.

1. According to Lemma 3.9, there is a B in $\tilde{\mathcal{S}}$ such that B_{τ_B} is the inverse of A_{τ_A}. Because of Lemma 3.10, we have $I \subset cl(S(B\boxtimes A)) \subset cl(A_{\tau_A}^{-1}S(A)) = cl(E_{\tau_A}(A))$ thus A coincides with A_{τ_A} on I.
2. Because of Lemma 3.10, we have $I = \cap_{A\in\mathcal{S}}cl(S(A))$. Fix $A \in \mathcal{S}$ and apply Proposition 3.4 4. for any $B \in \mathcal{S}$. This gives

   $$I \subset \cap_{B\in\mathcal{S}}cl(S(A\boxtimes B)) \subset A_{\tau_A}\left[\cap_{B\in\mathcal{S}}cl(S(B))\right] = A_{\tau_A}I,$$

   Therefore I is closed under the action of the inverses of the A_{τ_A}.

   Thanks to Lemma 3.9, it is closed under the action of the A_{τ_A} themselves. Together with the previous item, this concludes the proof for the $A \in \mathcal{S}$.

   Applying this to $\tilde{\mathcal{S}}$, we see that $I \cap \Sigma$ is closed under the action of the $A\tau$ for every $A \in \tilde{\mathcal{S}}$.
3. The third item follows from the famous Kakutani theorem, which we recall now.

   Theorem 3.12 (Kakutani [Kak38])**.** *If G is a group of uniformly continuous affine maps on a convex compact subset of a normed vector space, then all the elements of G have a common fixed point*

   According to the first two items and to Lemma 3.9, this theorem can be applied to the restriction of the $\tilde{A}$ to $K \cap \Sigma$. The common fixed point of the normalized matrices is a common eigenvector of the initial matrices.
4. First, let us notice that the function is well defined: τ_A only depends on $\tilde{A}$. The restrictions of the element of $\tilde{\mathcal{S}}$ to I are affine maps with the same common fixed point. Up to a change of coordinate, this point can be taken as the origin and the maps are equal to their linear parts. But the linear part of A_{τ_A} is the permutation of coordinates according to τ_A.

∎

4 Projection

In this section, we prove Theorem 1.1. $\mathcal{S}$ is a pseudo-primitive semigroup of projective maps to which we associate the following set $\tilde{\mathcal{S}} = \{A \in \mathbb{R}_{\max}^{d\times d} | \rho_{\max}(A) = 0, \overline{A} \in \mathcal{S}\}$. Notice that $\tilde{\mathcal{S}}$ is not necessary a semigroup because the product of two matrices with zero spectral radius do not necessary have zero spectral radius. In the previous section it was a semigroup because $\mathcal{S}$ was nice.

In this section and in the following one, we omit the notation $\boxtimes$ to shorten the formulas: AB means $A\boxtimes B$ and A^n means $A^{\boxtimes n}$.

Let P be a projectively bounded matrix in $\tilde{\mathcal{S}}$ such that $P^2 = P$. Let E have one element in each strongly connected component of $\mathcal{G}^c(P)$ and π be the matrix whose columns are the ones of P with columns numbers in E.

Take A a regular matrix in $\mathbb{R}_{\max}^{d\times d}$. Because of Proposition 2.2, for any $i \in E$, there are $\hat{A}_{ij} \in \mathbb{R}$ such that $PAP_{.i} = \boxplus_{j\in E} \hat{A}_{ij} \boxtimes P_{.j}$. In matrices notation, it is stated

$$PA\pi = \pi\hat{A}. \tag{4}$$

The following equations hold for any A_i in $\mathbb{R}_{\max}^{d\times d}$.

$$PA_1PA_2\cdots PA_n\pi = PA_1\cdots PA_{n-1}\pi\hat{A}_n = \pi\hat{A}_1\cdots\hat{A}_n,$$

which implies

$$\mathrm{Im}(PA_1\cdots PA_nP) = \pi\mathrm{Im}(\hat{A}_1\cdots\hat{A}_n). \tag{5}$$

Let $\hat{\mathcal{S}}$ be the **semigroup generated** by $\{\hat{A} | A \in \tilde{\mathcal{S}}\}$. The last equation implies that the images of the elements in $\hat{\mathcal{S}}$ are mapped by π onto images of elements of $\tilde{\mathcal{S}}$.

The essential lemma to deduce Theorem 1.1 from Theorem 3.1 is the following.

Lemma 4.1.

1. *If $\mathcal{S}$ is pseudo-primitive, then there is a projectively bounded $P \in \tilde{\mathcal{S}}$ such that $P^2 = P$ and $dim(\mathrm{Im}P) = \min_{A\in\mathcal{S}} dim(\mathrm{Im}A)$.*

2. *For such a P, $\hat{\mathcal{S}}$ is **nice**.*

Proof.

1. let $B \in \tilde{\mathcal{S}}$ be such that $dim(\mathrm{Im}B) = \min_{A\in\tilde{\mathcal{S}}} dim(\mathrm{Im}A)$. Since $\mathcal{S}$ is pseudo-primitive, there is a projectively bounded $C \in \tilde{\mathcal{S}}$. Now CB is projectively bounded and $dim(\mathrm{Im}CB) = \min_{A\in\tilde{\mathcal{S}}} dim(\mathrm{Im}A)$. According to Corollary 2.3, there is a power D of CB, such that $P = \tilde{D}$ satisfies $P^2 = P$ and

$$dim(\mathrm{Im}P) = dim(\mathrm{Im}D) \le dim(\mathrm{Im}CB) \le \min_{A\in\tilde{\mathcal{S}}} dim(\mathrm{Im}A)$$

but because of minimality, the inequality is an equality.

2. By construction all the entries of the elements of $\hat{\mathcal{S}}$ are finite. If $\hat{\mathcal{S}}$ where not **nice**, then there would be an element in $A \in \hat{\mathcal{S}}$ with $dim(\mathrm{Im}A)$ strictly less than the cardinality of E, which is also the dimension of $\mathrm{Im}P$ according to Lemma 3.3 and is equal to $\min_{A\in\tilde{\mathcal{S}}} dim(\mathrm{Im}A)$ by definition. It would imply the existence of $A_1 \cdots A_n$ in $\tilde{\mathcal{S}}$ such that $\mathrm{Im}(PA_1 \cdots PA_nP) = \pi\mathrm{Im}(\hat{A}_1 \cdots \hat{A}_n)$ also has dimension strictly less than $\min_{A\in\tilde{\mathcal{S}}} dim(\mathrm{Im}A)$, which is a contradiction.

□

Proof of Theorem 1.1. We apply Lemma 4.1 to get a nice semigroup $\hat{\mathcal{S}}$. Then, Theorem 3.1 gives an $\hat{x} \in \mathbb{R}^E$ and such that $B\boxtimes\hat{x} = \hat{x}$ for any $B \in \hat{\mathcal{S}}$. Setting $x_0 = \pi\hat{x}$, Equation (4), says that $PAx_0 = x_0$ for any $A \in \tilde{\mathcal{S}}$, that is $\overline{P}A\overline{x_0} = \overline{x_0}$ for any $A \in \mathcal{S}$

The next step consists in proving that

$$\pi(\cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}\hat{A}) = \cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}PA. \tag{6}$$

Equation (5) implies $\cap_{\overline{A}\in\mathcal{S}}\pi(\mathrm{Im}\hat{A}) = \cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}PA$ thus $\pi(\cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}\hat{A}) \subset \cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}PA$. Let us prove the converse inclusion.

For any $\epsilon > 0$, Lemma 3.10 4. gives a $C \in \hat{\mathcal{S}}$ such that $\mathrm{Im}C \subset [\cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}\hat{A}]^\epsilon$. Since π is 1-Lipschitz, we have

$$\cap_{\overline{A}\in\mathcal{S}} \mathrm{Im}PA \subset \mathrm{Im}\pi C \subset (\pi[\cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}\hat{A})])^\epsilon. \tag{7}$$

Since $\pi[\cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}\hat{A}) = \mathbb{R}\boxtimes\pi[\cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}\hat{A}) \cap \Sigma]$ is closed as the product of $\mathbb{R}$ and the compact space $\pi[\cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}\hat{A}) \cap \Sigma]$, letting ϵ tend to 0 in Equation (7) concludes the proof of Equation (6).

On $\mathrm{Im}P$, any map PA with $A \in \mathcal{S}$ is given by some $\hat{A} \in \hat{\mathcal{S}}$ which satisfy Equation (4). The restriction of $\hat{A}$ to $\cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}\hat{A}$ is an element of a finite group, so that it has finite order. Therefore the restriction of PA to $\cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}PA$ also has finite order, which implies that the set of all these restrictions is a group. It is finite because the set of all possible restrictions of $\hat{A} \in \hat{\mathcal{S}}$ is finite.

Finally, the restrictions of the PA to $\cap_{\overline{A}\in\mathcal{S}}\mathrm{Im}PA$ are isometries, because they are 1-Lipschitz an so is their inverse. □

5 Central limit theorem

In this section, we prove Corollary 1.2.

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random regular matrices that take countably many values such that the semigroup $\mathcal{S}$ of projective maps generated by those values is pseudo-primitive. Let γ be the top Lyapunov exponent of $(A_n)_{n\in\mathbb{N}}$.

In [Mer07], we proved that the sequence $\left(\frac{1}{\sqrt{n}}\left(A_n \cdots A_0 0 - n\gamma\mathbf{1}\right)\right)_{n\in\mathbb{N}}$ converges in law to a normal distribution with dimension 1 under the additional assumption that there is some $N \in \mathbb{N}$ such that $\overline{A_N \cdots A_0}$ is a constant with positive probability.

Theorem 1.1 applied to $\mathcal{S}$ gives a projective map P in $\mathcal{S}$ and a vector $x_0 \in \mathbb{R}^d$ whose projective image is a fixed point of $P\mathcal{S}$. Therefore the restriction of P to the orbit of x_0 under the action of $\mathcal{S}$ is a constant. By definition of $\mathcal{S}$, there is some $n \in \mathbb{N}$ such that $\overline{A_n \cdots A_0} = P$ with positive probability. The proof of [Mer07] can therefore be adapted to prove the convergence of $\left(\frac{1}{\sqrt{n}}\left(A_n \cdots A_0 x_0 - n\gamma \mathbf{1}\right)\right)_{n\in\mathbb{N}}$, once we noticed that $\left(\max_i \left(A_n \cdots A_0 x_0 - x_0\right)_i\right)_{n\in\mathbb{N}}$ is a subadditive sequence and that $\max_i \left(Au - x_0\right)_i - \max_i \left(u - x_0\right)_i$ only depends on A and $\overline{u}$.

Since max-plus maps are nonexpansive, the convergence of $\left(\frac{1}{\sqrt{n}}\left(A_n \cdots A_0 x_0 - n\gamma \mathbf{1}\right)\right)_{n\in\mathbb{N}}$ implies the convergence of $\left(\frac{1}{\sqrt{n}}\left(A_n \cdots A_0 X_0 - n\gamma \mathbf{1}\right)\right)_{n\in\mathbb{N}}$ for any initial condition X_0.

Thanks

This article is based on my work as a post-doctoral fellow in the ANR project MASED (06-JCJC-0069), headed by J. Mairesse. I gratefully thank P. Butkovic and H. Schneider for inviting me to present these results at the minisymposium on max-plus agebra they organized at ILAS 2008, as well as S. Sergeev for useful suggestions.

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