# On Nevanlinna–Djrbashian Square Summable Classes in the Unit Disc of the Complex Plane

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**Abstract**

Some, arbitrarily wide, weighted classes of square summable subharmonic functions are considered in the unit disc of the complex plane. It is proved that any function of such a class is representable as the sum of some Green type potential and a Poisson type integral, and these summands are orthogonal in a suitable Lebesgue space $L^2_\omega$ with rotation-invariant measure. These representations particularly become factorizations for all functions holomorphic in the unit disc.

## 0.1 Introduction

The results given below are some continuation of the series of M. M. Djrbashian investigations that resulted in his factorization theory of functions meromorphic in the unit disc of the complex plane [1, 5, 6, 7]. Note that some applications of the results of [7], along with the author's theory in the half plane [11], are given in [8, 9, 11]. More precisely, the below results continue author's work [10] particularly generalizing M. M. Djrbashian's early results [2, 3] devoted to analysis of some spaces and classes of regular functions defined by means of the Riemann–Liouville fractional integration. Namely the spaces $A^p_\alpha$, $\alpha > -1$, $p \geq 1$ (or initially $H_p(\alpha)$), of functions $f(z)$ holomorphic in $|z| < 1$, which are defined by the condition

$$\iint_{|\zeta|<1}(1-|\zeta|)^\alpha |f(\zeta)|^p d\sigma(\zeta) = \sup_{0<r<1}\int_0^1 (1-t)^\alpha t dt \int_0^{2\pi} |f(tre^{i\vartheta})|^p d\vartheta < +\infty,$$

where $\sigma(\zeta)$ is Lebesgue's area measure, and Nevanlinna's classes of functions $f(z)$ meromorphic in $|z| < 1$ (see [1], Section 216), which are defined by the condition

$$\int_0^1 (1-r)^\alpha T(r,f)dr = \sup_{0<r<1}\int_0^1 (1-t)^\alpha T(rt,f)dt < +\infty, \tag{1}$$

($\alpha > -1$), where $T(r,f)$ is Nevanlinna's growth characteristic. Note that for functions $f(z)$ holomorphic in $|z| < 1$, the condition (1) is equivalent to

$$\iint_{|z|<1} \big|\log|f(z)|\big|(1-|z|)^\alpha d\sigma(z) < +\infty. \tag{2}$$

In difference with this, here we consider some classes of functions $u(z)$ subharmonic in $|z| < 1$ (which in particular can be $\log|f(z)|$ with $f(z)$ holomorphic in $|z| < 1$), the *squares* of which are summable as in (2), *with some general measure*, and the union of the considered classes coincides with the set of all functions subharmonic in $|z| < 1$.

## 0.2 Classes of square summable subharmonic functions

We assume that $\omega(x) \in \Omega_{N^2}$, i.e. $\omega(x)$ is a continuously differentiable in $[0,1)$, strictly decreasing, real function, such that $\omega(0) = 1$, $\omega(1) = \omega(1-0) = 0$ and $|\omega'(x)|$ is strictly decreasing in $[0,1)$. Further, we define $N^2_\omega$ as the set of functions $u(z)$ subharmonic in $|z| < 1$, such that $u(z)$ belongs to the Lebesgue space $L^2_\omega$ considered in [10], i.e.

$$\|u\|^2_{L^2_\omega} = \frac{1}{2\pi}\iint_{|z|<1}\big[u(z)\big]^2 d\mu_\omega(z) < +\infty, \tag{3}$$

where $d\mu_\omega(re^{i\vartheta}) = -d\vartheta d\omega(r^2)$. First, note that the following statement is true.

**Proposition.** *The union of the classes $N^2_\omega$ over all $\omega(x) \in \Omega_{N^2}$ coincides with the set of all functions subharmonic in $|z| < 1$.*

Supposing that $d_0 \in (0,1)$ is some fixed number, introduce the following Green potential formed by the ordinary Blaschke factors and a bounded, nonnegative Borel measure $\nu(\zeta)$ in $|\zeta| \le d_0$:

$$G_0(z) = \iint_{|\zeta|<d_0} \log |b(z,\zeta)|\, d\nu(\zeta), \quad b(\lambda,\zeta) = \frac{\zeta - \lambda}{1 - \lambda\overline{\zeta}} \frac{|\zeta|}{\zeta}.$$

Then $G_0(z) \in N^2_\omega$, and hence, forming $G_0(z)$ by the Riesz associated measure of a function $u(z) \in N^2_\omega$ we conclude that also the subharmonic function $u_0(z) = u(z) - G_0(z)$ is of $N^2_\omega$. So, further we shall assume that the Riesz associated measure of a function $u(z) \in N^2_\omega$ is such that

$$\inf \left\{ |\zeta| : \zeta \in \operatorname{supp} \nu \right\} \ge d_0,$$

where $d_0 \in (0,1)$ is a fixed number, and this assumption will not affect the generality of our argument.

Further, one can see that $L^2_\omega \in L^1_\omega$ for any $\omega(t) \in \Omega_{N^2}$, and consequently the inclusion $u(z) \in N^2_\omega$ implies $u(z) \in L^1_\omega$. Hence, by Theorem 4.3 of [10], the Riesz associated measure of $u(z)$ satisfies the density condition

$$\iint_{|\zeta|<1} \left( \int_{|\zeta|^2}^1 \omega(t) dt \right) d\nu(\zeta) < +\infty, \tag{4}$$

and the following Riesz type representation is true:

$$u(z) = G(z) + U(z), \quad |z| < 1, \tag{5}$$

where

$$G(z) = \iint_{|\zeta|<1} \log |b_\omega(z,\zeta)| d\nu(\zeta)$$

is a Green type potential formed by the Blaschke type factors $b_\omega(z,\zeta)$ of [10]. This potential is convergent in $|z| < 1$ in virtue of (4), and the integral

$$\begin{aligned} U(z) &= \frac{1}{\pi} \iint_{|\zeta|<1} u(\zeta) \operatorname{Re} \left\{ C_\omega(z\overline{\zeta}) \right\} d\mu_\omega(\zeta) - u(0) \\ &= \frac{1}{\pi} \iint_{|\zeta|<1} \left[ u(\zeta) - \frac{u(0)}{2} \right] \operatorname{Re} \left\{ C_\omega(z\overline{\zeta}) \right\} d\mu_\omega(\zeta), \end{aligned}$$

where

$$C_\omega(z) = \sum_{k=0}^{\infty} \frac{z}{\Delta_k}, \quad \Delta_k = -\int_0^1 t^k d\omega(t),$$

is M. M. Djrbashian's Cauchy type kernel, is uniformly convergent in $|z| < 1$, where it represents a harmonic function.

The representation (5) is inherent to a class of functions which is wider than $N^2_\omega$. Thus, the inclusion $u(z) \in N^2_\omega$ implies some additional statements. Particularly, the given below two theorems are true.

**Theorem 1.** *The following statements are true:*

1°. *Both summands $G(z)$ and $U(z)$ in the right-hand side of the representation (5) of a function $u(z) \in N^2_\omega$ ($\omega \in \Omega_{N^2}$) are of $N^2_\omega$.*

2°. *The operator*

$$Q_\omega u(z) = \frac{1}{\pi} \iint_{|\zeta|<1} \left[ u(\zeta) - \frac{u(0)}{2} \right] \mathrm{Re}\left\{ C_\omega(z\overline{\zeta}) \right\} d\mu_\omega(\zeta)$$

*is identical on the set of harmonic functions of $N^2_\omega$ and it takes any Green type potential $G(z) \in N^2_\omega$ to identical zero.*

3°. *If $U(z)$ is any harmonic function of $N^2_\omega$, then in $L^2_\omega$ the function $U(z) - U(0)$ is orthogonal to any Green type potential $G(z) \in N^2_\omega$.*

**Theorem 2.** *If $\nu(\zeta)$ is a nonnegative Borel measure in $|\zeta| < 1$, which satisfies (4) with some $\omega(x) \in \Omega_{N^2}$, then*

$$\|G\|^2_{L^2_\omega} = \iint_{|\zeta|<1} \iint_{|\zeta'|<1} \big( \log|b_\omega(z,\zeta)|, \log|b_\omega(z,\zeta')| \big)_\omega \, d\nu(\zeta) d\nu(\zeta'), \qquad (6)$$

*where the left- and the right-hand sides are finite or infinite simultaneously.*

In virtue of Theorems 1 and 2, the class $N^2_\omega$ coincides with the set of all those functions $u(z)$ subharmonic in $|z| < 1$, which are representable in the form (5), where $U(z)$ is a harmonic function of $N^2_\omega$ and $G(z)$ is a convergent Green type potential with a Riesz measure for which the right-hand side expression of (6) is finite. Note that using some formulas from [10] the value of the inner product in that expression can be exactly calculated. This results in a density condition for the measure $d\nu(\zeta)d\nu(\zeta')$, which accepts the arguments of $\zeta$ and $\zeta'$.

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