Representations for the generalized Drazin inverse of additive perturbations

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Let $\mathcal{B}$ be a unital complex Banach algebra. An element $a \in \mathcal{B}$ is said to have a *generalized Drazin inverse* if there exists $x \in \mathcal{B}$ such that

$$xa = ax, \quad x = ax^2, \quad a - a^2x \text{ is quasinilpotent.}$$

In this case, the generalized Drazin inverse of a is unique and is denoted by a^D. If in the previous definition $a - a^2x$ is in fact nilpotent then a^D is the conventional *Drazin inverse* of a. It is well known that if a and b have generalized Drazin inverse and $ab = ba = 0$, then $(a+b)^D = a^D + b^D$. This result was generalized in [Djordjević and Wei, Additive result for the generalized Drazin inverse, J. Austral. Math. Soc. 73 (2002) 115-125] under the one side condition $ab = 0$. Recently, in [Castro and Koliha, New Additive results for the g-Drazin inverse, Proc. Roy. Soc. Edinburgh Sect. A 134 (2005) 657-666], [Cvetković-Ilić *et al.*, Additive results for the generalized Drazin inverse in a Banach algebra, Linear Algebra Appl. 418 (2006) 53-61], weaker conditions were given under which $(a+b)^D$ could be explicitly expressed in terms of a, a^D, b, and b^D.

In this paper we study the generalized Drazin inverse of the sum $a + b$, where the perturbation b is a quasinilpotent element, and we obtain a representation for $(a+b)^D$ under new conditions which relax the condition $ab = 0$. Our approach is based on a representation for the resolvent of a 2×2 matrix with entries in a Banach algebra, which we provide, and the Laurent expansion of the resolvent in terms of the generalized Drazin inverse. Our results can be applied to obtain different representations of the generalized Drazin inverse of block matrices

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

under certain conditions, in terms of the individual blocks. In particular, we can write M as the sum of a block triangular matrix and a nilpotent matrix and apply the additive perturbation result given to obtain a representation for M^D. It extends the result of Meyer and Rose for the Drazin inverse of

a block triangular matrix. Finally, we present a numerical example for the Drazin inverse of 2×2 block matrices over the complex numbers.

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