A fundamental paradox in learning algebra

The generalizing, formalizing and unifying nature of some of the concepts of Linear Algebra leads to a high level of abstraction, which in turn constitutes a source of difficulties for students. When asked to deal with new expressions, new symbolism and rules of calculation, students face what researchers in mathematics education — such as Dorier, Rogalski, Sierpinska or Harel — have identified as 'the obstacle of formalism'.

Teachers bring in new mathematical objects, sometimes in a non explicit way, by using at once the symbols referring to these objects or to the related relations, without explaining or justifying the meaning or the relevance of their choices, regarding this new symbolism. Calculations and manipulations with these new objects build up to new algebras (vector or matrix algebras) more complex than basic (high school) algebra, but nevertheless syntactically modelled on it. The gap thus caused reveals itself when students bring out inconsistent or meaningless writings : "The obstacle of formalism manifests itself in students who operate at the level of the form of expressions without seeing these expressions as referring to something other than themselves. One of the symptoms is the confusion between categories of mathematical objects ; for example, sets are treated as elements of sets, transformations as vectors, relations as equations, vectors as numbers, and so on" (Sierpinska et al., 1999, p. 12). For too many students attending their first course in Linear Algebra, the latter is nothing but a catalogue of very abstract notions, for which they have almost no understanding, being overwhelmed by a flood of new words, new symbols, new definitions and new theorems (Dorier, 1997).

Our talk will be based on a study conducted within the context of a master degree in mathematics education (maîtrise en didactique des mathématiques, Université du Québec à Montréal ; cf. Corriveau & Tanguay, 2007). Through this study, we tried to have a better understanding of transitional difficulties, due to the abrupt increase in what is expected from students with respect to formalism and proof, when going from Secondary schools to 'Cegeps' (equivalent in Québec of 'upper secondary' or 'high-school', 17-19 years of age). The Linear Algebra courses having been identified as those in which such transitional problems are the most acute, we first selected, among all problems submitted in a given L. A. course — the teacher of which was ready to participate in the study — those involving a proof or a reasoning at least partly deductive.

Through the systematic analysis of these problems, we evaluated and compared their level of difficulty, as well as students' preparation for coping with such difficulties, from an 'introduction-to-formalism' perspective. The framework used to analyse the problems stemmed from a remodelling of Robert's framework (1998). The remodelling was a consequence of having compared/confronted an *a priori* analysis of three problems (using Robert's framework), with the analysis of their erroneous solutions as they appeared in twelve students' homework copies.

Among the conclusions brought up by the study, we shall be interested in the following ones:

- Mathematical formalism allows a 'compression' of the mathematical discourse, simplification and systematization of the syntax, by which one operates on this discourse with better efficiency. But this improvement in efficiency is achieved to the detriment of meaning. As in Bloch and al. (2007), the study confirms that "...formal written discourse does not carry *per se* the meaning of neither the laws that it states nor the objects that it sets forth." For many students, symbolic manipulations are difficult in Linear Algebra because meaning has been lost somewhere. By trying to have a better understanding of the underlying obstacle, we came

to identify what we call 'the fundamental paradox in learning [a new] algebra', some elements of which will be discussed further in the talk.

- The analysis of students' written productions brings us to observe that in the process of proving, difficulties caused by the introduction of new objects and new rules of calculation on the one hand, and difficulties related to controlling the deductive reasoning and its logical structure on the other, are reinforcing one another.
- A better understanding of students' errors, by an error-analysis such as the one done in the study, allows a better evaluation of the difficulty level of what is asked to students, and thus a better understanding of the problems linked to academic transitions (from lower-secondary to upper-secondary to university) in mathematics. Such analyses could give Linear Algebra teachers better tools, for estimating the difficulties in the tasks they submit to their students, as well as for understanding the underlying cognitive gaps and ruptures. It would be advisable that teachers be introduced to such error-analysis work, in the setting of their pre-service or in-service instruction.

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