Solving two-sided (max,plus)-linear equation systems.

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Systems of equations of the following form will be considered:

$$a_i(x) = b_i(x) \ i \in I, \tag{1}$$

where $I = \{1, \ldots, m\}, \ J = \{1, \ldots, m\}$,

$$a_i(x) = max_{j \in J}(a_{ij} + x_j), \ b_i(x) = max_{j \in J}(b_{ij} + x_j) \ \ \forall i \in I$$

and $a_{ij}, \ b_{ij}$ are given real numbers.
The aim of the contribution is to propose a polynomial method for solving system (1). Let M be the set of all solutions of (1), let $M(\overline{x})$ denote the set of solutions of system (1) satisfying the additional constraint $x \leq \overline{x}$, where $\overline{x}$ is a given fixed element of R^n. The proposed method either finds the maximum element of the set $M(\overline{x})$ (i.e. element $\hat{x} \in M(\overline{x})$, for which $x \in M(\overline{x})$ implies $x \ \leq \hat{x}$), or finds out that $M(\overline{x}) = \emptyset$. The results are based on the following properties of system (1) (to simplify the notations we will assume in the sequel w.l.o.g. that $a_i(\overline{x}) \geq b_i(\overline{x}) \ \ \forall \ \ i \in I$ and $\overline{x} \notin M(\overline{x})$):

(i) $M(\overline{x}) \ = \ \emptyset \ \Rightarrow M \ = \ \emptyset$.
(ii) Let $K_i = \{k \in J \ ; \ a_{ik} \leq b_{ik}\} \ \forall i \in I$. If for some $i_0 \in I$ the set $K_{i_0} = \emptyset$, then $M(\overline{x}) = \emptyset$.
(iii) Let $\beta_i(\overline{x}) = max_{k \in K_i}(b_{ik} + \overline{x}_k)$, $L_i(\overline{x}) = \{j \in J \ ; \ a_{ij} + \overline{x}_j \ > \ \beta_i(\overline{x})\}$, $\forall \ i \in I$. If $\bigcup_{i \in I} L_i(\overline{x}) = J$, then $M(\overline{x}) = \emptyset$.
(iv) Let $V_j(\overline{x}) = \{i \in I; j \in L_i(\overline{x})\}$, let $\overline{x}_j^{(1)} = min_{i \in V_j(\overline{x})}(\beta_i(\overline{x}) - a_{ij})$ for all $j \in J$, for which $V_j(\overline{x}) \neq \emptyset$ and $\overline{x}_j^{(1)} = \overline{x}_j$ otherwise. Let $\beta_i(\overline{x}^{(1)}) \ < \ \beta_i(\overline{x})$ for all $i \in I$. Then for at least one $i \in I$ the value $\beta_i(\overline{x}^{(1)})$ is equal to at least one of the threshold values $b_{ij} + \overline{x}_j \ < \ \beta_i(\overline{x})$.

The method successively determines variables, which have to be decreased if equalities in (1) should be reached. If all variables have to be set in movement, no solution of (1) exists. If the set of unchanged variables is nonempty, the maximum element of (1) is obtained. Using these properties a polynomial behavior of the proposed method can be proved (in case of rational or integer inputs). Possibilities of further generalizations and usage in optimization with constraints (1), as well as applications to synchronization problems will be briefly discussed.