Orderings of matrix algebras and their applications

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The full matrix algebra $M_n(\mathbf{F})$ over a totally-ordered subfield $\mathbf{F}$ of the reals becomes a *partially ordered algebra* by a partial order relation $\leq$ on the set $M_n(\mathbf{F})$, if for any $A, B, C \in M_n(\mathbf{F})$ from $A \leq B$ it follows that:

(1) $A + C \leq B + C$

(2) if $C \geq 0$ then $AC \leq BC$ and $CA \leq CB$

(3) if $\mathbf{F} \ni \alpha \geq 0$ then $\alpha A \leq \alpha B$.

Our interest is when the order $\leq$ is a lattice or at least is directed. Then we have a *lattice-ordered algebra of matrices* or a *directly-ordered algebra of matrices*. Those concepts originate in 1956 in Birkhoff and Pierce in [1]. The first example of a lattice-ordered algebra of matrices is, of course, with the *usual* entry-wise ordering. In this ordering the identity matrix I is positive. In 1966 E. Weinberg proved in [6] that the positivity of I forces a lattice-ordering to be (isomorphic to) the usual one in $M_2(\mathbf{F})$ and conjectured the same for all $n \geq 2$. The conjecture was positively solved in 2002 by J. Ma and P. Wojciechowski in [4]. The proof involved a *cone-theoretic* approach, by first establishing existence of a P-invariant cone O in $\mathbf{F}^n$, i.e. satisfying the condition that for every matrix $M \in P$, $M(O) \subseteq O$, where P is the *positive cone* of the ordering $\leq$ ($P = \{A \in M_n(\mathbf{F}) : A \geq 0\}$.) With help of compactness of a unit sphere in $\mathbf{R}^n$ and the Zorn's Lemma, we obtained all the desired properties of the cone O that led us to the conclusion of the conjecture.
The first part of the talk will briefly outline the method.

The above considerations allowed us to comprehensively describe all lattice orders of $M_n(\mathbf{F})$ (J. Ma and P. Wojciechowski [5]): the algebra $M_n(\mathbf{F})$ is lattice-ordered (within an isomorphism) if and only if

$$A \geq 0 \Leftrightarrow A = \sum_{i,j=1}^{n} \alpha_{ij} E_{ij} H^T$$

with

$$\alpha_{ij} \geq 0$$

$i, j = 1, \ldots, n$, for some given H nonsingular with nonnegative entries and E_{ij} having 1 in the ij entry and zeros elsewhere.

As a first application, we will describe all *multiplicative bases* in the matrix algebra $M_n(\mathbf{F})$ and provide their enumeration for small n (C. De La Mora and P. Wojciechowski 2006 [2].) In a finite-dimensional algebra over a field $\mathbf{F}$, a basis $\mathfrak{B}$ is called *a multiplicative basis* provided that $\mathfrak{B} \cup \{0\}$ forms a semigroup. Although these bases (endowed with some additional algebraic properties) have been studied in the representation theory, they lacked a comprehensive classification for matrix algebras. The first example of a multiplicative basis of $M_n(\mathbf{F})$ should of course be $\{E_{ij}, i, j = 1, \ldots, n\}$. Every lattice order on $M_n(\mathbf{F})$ corresponds to a nonsingular $n \times n$ matrix H with nonnegative entries. It turns out that if the entries are either 0 or 1, the basic matrices resulting in the definition of the lattice order, i.e. the matrices $E_{ij}H^T$ form a multiplicative basis, and conversely, every multiplicative basis corresponds to a nonsingular zero-one matrix. After identification of the isomorphic semigroups and also identification of the matrices that have just permuted rows and columns, the above correspondence is one-to-one. The number of zero-one nonsingular matrices, although lacking a formula so far, is known for a few small n values. This, together with the conjugacy class method from group theory, allowed us to calculate the number of nonequivalent multiplicative bases up to dimension 5: 1, 2, 8, 61, 1153.

Another application concerns certain directed partial orders of matrices that appear naturally in linear algebra and its applications. It is related to the research of matrices preserving cones, established in the seventies, among others by R. Loewy and H. Schneider in [3]. Besides the lattice orders (corresponding to the simplicial cones), the best studied ones are the orders whose positive cones are the sets $\Pi(O)$, of all matrices preserving a regular (or full) cone O in an n-dimensional Euclidean space. It can be shown that O is essentially the only $\Pi(O)$-invariant cone (P. Wojciechowski [7].) Consequently, we obtain a characterization of all maximal directed partial orders on the

$n \times n$ matrix algebra: a directed order is maximal if and only if its positive cone P satisfies $P = \Pi(O)$ for some regular cone O. The method used in the proof involves a concept of *simplicial separation*, allowing a regular cone to be separated from an outside point by means of a simplicial cone.

Some open questions related to the discussed topics will be raised during the talk.

References

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