Numerical methods for two-parameter eigenvalue problems

By *Bor Plestenjak*. We consider the *two-parameter eigenvalue problem* [1]

$$\begin{aligned} A_1x_1 &= \lambda B_1x_1 + \mu C_1x_1, \\ A_2x_2 &= \lambda B_2x_2 + \mu C_2x_2, \end{aligned} \tag{1}$$

where A_i, B_i, and C_i are given $n_i \times n_i$ matrices over $\mathbb{C}$, $\lambda, \mu \in \mathbb{C}$, and $x_i \in \mathbb{C}^{n_i}$ for $i = 1, 2$. A pair (λ, μ) is an *eigenvalue* if it satisfies (1) for nonzero vectors x_1, x_2. The tensor product $x_1 \otimes x_2$ is then the corresponding *eigenvector*. On the tensor product space $S := \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ of the dimension $N := n_1n_2$ we can define *operator determinants*

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2, \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2. \end{aligned}$$

The two-parameter problem (1) is *nonsingular* if its operator determinant Δ_0 is invertible. In this case $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute and problem (1) is equivalent to the associated problem

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z, \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned} \tag{2}$$

for decomposable tensors $z \in S$, $z = x_1 \otimes x_2$. Some numerical methods and a basic theory of the two-parameter eigenvalue problems will be presented. A possible approach is to solve the associated couple of generalized eigenproblems (2), but this is only feasible for problems of low dimension because the size of the matrices of (2) is $N \times N$. For larger problems, if we are interested in a part of the eigenvalues close to a given target, the Jacobi–Davidson method [3, 4, 5] gives very good results. Several applications lead to singular two-parameter eigenvalue problems where Δ_0 is singular. Two such examples are model updating [2] and the quadratic two-parameter eigenvalue problem

$$\begin{aligned} (S_{00} + \lambda S_{10} + \mu S_{01} + \lambda^2 S_{20} + \lambda\mu S_{11} + \mu^2 S_{02})x &= 0 \\ (T_{00} + \lambda T_{10} + \mu T_{01} + \lambda^2 T_{20} + \lambda\mu T_{11} + \mu^2 T_{02})y &= 0. \end{aligned} \tag{3}$$

We can linearize (3) as a singular two-parameter eigenvalue problem, a possible linearization is

$$\left(\begin{bmatrix} S_{00} & S_{10} & S_{01} \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & S_{20} & \frac{1}{2}S_{11} \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & \frac{1}{2}S_{11} & S_{02} \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}\right) \widetilde{x} = 0$$

$$\left(\begin{bmatrix} T_{00} & T_{10} & T_{01} \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & T_{20} & \frac{1}{2}T_{11} \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & \frac{1}{2}T_{11} & T_{02} \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}\right) \widetilde{y} = 0,$$

where $\widetilde{x} = \begin{bmatrix} x \\ \lambda x \\ \mu x \end{bmatrix}$ and $\widetilde{y} = \begin{bmatrix} y \\ \lambda y \\ \mu y \end{bmatrix}$. Some theoretical results and numerical methods for singular two-parameter eigenvalue problems will be presented.

References

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[3] M. E. Hochstenbach and B. Plestenjak, *A Jacobi–Davidson type method for a right definite two-parameter eigenvalue problem*, SIAM J. Matrix Anal. Appl., 24 (2002), pp. 392–410.

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[5] M. E. Hochstenbach and B. Plestenjak, *Harmonic Rayleigh–Ritz extraction for the multiparameter eigenvalue problem*, to appear in ETNA.