The class of Inverse-Positive matrices with cheker-board pattern

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In economics as well as other sciences, the inverse-positivity of real square matrices has been an important topic. A nonsingular real matrix A is said to be inverse-positive if all the elements of its inverse are nonnegative. An inverse-positive matrix being also a Z-matrix is a nonsingular M-matrix, so the class of inverse-positive matrices contains the nonsingular M-matrices, which have been widely studied and whose applications, for example, in iterative methods, dynamic systems, economics, mathematical programming, etc, are well known. Of course, every inverse-positive matrix is not an M-matrix. For instance,

$$A = \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$$

is an inverse-positive matrix that is not an M-matrix. The concept of inverse-positive is preserved by multiplication, left or right positive diagonal multiplication, positive diagonal similarity and permutation similarity. The problem of characterizing inverse-positive matrices has been extensively dealt with in the literature (see for instance [1]). The interest of this problem arises from the fact that a linear mapping $F(x) = Ax$ from R^n into itself is inverse issotone if and only if A is inverse-positive. In particular, this allows us to ensure the existence of a positive solution for linear systems $Ax = b$ for any $b \in R^n_+$. In this paper we present several matrices that very often occur in relation to systems of linear or nonlinear equations in a wide variety of areas including finite difference methods for contour problems, for partial differential equations, Leontief model of circulating capital without joint production, and Markov processes in probability and statistics. For example, matrices that for size 5×5 have the form

$$A = \begin{pmatrix} 1 & -a & 1 & -a & 1 \\ 1 & 1 & -a & 1 & -a \\ -a & 1 & 1 & -a & 1 \\ 1 & -a & 1 & 1 & -a \\ -a & 1 & -a & 1 & 1 \end{pmatrix},$$

where a is a real parameter with economic interpretation. Are these matrices inverse-positive?. We study the answer of this question and we analyze when the concept of inverse-positive is preserved by the Hadamard product $A \circ A^{-1}$. In this work we present some conditions in order to obtain new characterizations for inverse-positive matrices. Johnson in [3] studied the possible sign patterns of a matrix which are compatible with inverse-positiveness. Following his results we analyze the inverse-positive concept for a particular type of pattern: the chekecboard pattern. An $n \times n$ real matrix $A = (a_{i,j})$ is said to have a checkerboard pattern if $\text{sign}(a_{i,j}) = (-1)^{i+j}$, $i, j = 1, 2, \ldots, n$. We study in this paper the inverse-positivity of bidiagonal, tridiagonal and lower (upper) triangular matrices with checkerboard pattern. We obtain characterizations of the inverse-positivity for each class of matrices. Several authors have investigated about the Hadamard product of matrices. Johnson [2] showed that if the sign pattern is properly adjusted the Hadamard product of M-matrices is again an M-matrix and for any pair M,N of M-matrices the Hadamard product $M \circ N^{-1}$ is again an M-matrix. This result does not hold in general for inverse-positive matrices. We analyze when the Hadamard product $M \circ N^{-1}$, for M,N checkerboard pattern inverse-positive matrices , is an inverse-positive matrix.

References

[1] A. Berman, R.J. Plemmons, *Nonnegative matrices in the Mathematical Sciences*, SIAM 1994.

[2] C.R. Johnson, *A Hadamard Product Involving M-matrices*, Linear Algebra and its Applications, 4 (1977) 261-264.

[3] C.R. Johnson, *Sign patterns of inverse nonnegative matrices*, Linear Algebra and its Applications, 55 (1983) 69-80.