Model Order Reduction of Systems with Coupled Parameters

By *Peter Benner and Lihong Feng*

We consider model order reduction of parametric systems with parameters which are nonlinear functions of the frequency parameter s. Such systems result from, for example, the discretization of electromagnetic systems with surface losses [1]. Since the parameters are functions of the frequency s, they are highly coupled with each other. We see them as individual parameters when we implement model order reduction. By analyzing existing methods of computing the projection matrix for model order reduction, we show the applicability of each method and propose an optimized method for the parametric system considered in this paper. The transfer function of the parametric systems considered here take the form

$$H(s) = sB^{\mathrm{T}}(s^2 I_n - 1/\sqrt{s}D + A)^{-1}B, \tag{1}$$

where A, D and B are $n \times n$ and $n \times m$ matrices, respectively, and I_n is the identity of suitable size. To apply parametric model order reduction to (1), we first expand $H(s)$ into a power series. Using a series expansion about an expansion point s_0, and defining $\sigma_1 := \frac{1}{s^2\sqrt{s}} - \frac{1}{s_0^2\sqrt{s_0}}$, $\sigma_2 := \frac{1}{s^2} - \frac{1}{s_0^2}$, we may use the three different methods below to compute a projection matrix V and get the reduced-order transfer function

$$\hat{H}(s) = s\hat{B}^{\mathrm{T}}(s^2 I_r - 1/\sqrt{s}\hat{D} + \hat{A})^{-1}\hat{B},$$

where $\hat{A} = V^T A V$, $\hat{B} = V^T B$, etc., and V is an $n \times r$ projection matrix with $V^T V = I_r$. To simplify notation, in the following we use $G := I - \frac{1}{s_0^2\sqrt{s_0}}D + \frac{1}{s_0^2}A$, $B_M := G^{-1}B$, $M_1 := G^{-1}D$, and $M_2 := -G^{-1}A$.

Directly computing V

A simple and direct way for obtaining V is to compute the coefficient matrices in the series expansion

$$\begin{aligned} H(s) \;=\; & \tfrac{1}{s}B^{\mathrm{T}}[B_M + (M_1 B_M \sigma_1 + M_2 B_M \sigma_2) + (M_1^2 B_M \sigma_1^2 \\ & +(M_1 M_2 + M_2 M_1)B_M \sigma_1 \sigma_2 + M_2^2 B_M \sigma_2^2) + (M_1^3 B_M \sigma_1^3 + \ldots) + \ldots], \end{aligned} \tag{2}$$

by direct matrix multiplication and orthogonalize these coefficients to get the matrix V [2]. After the coefficients B_M, M_1B_M, M_2B_M, $M_1^2B_M$, $(M_1M_2 + M_2M_1)B_M$, $M_2^2B_M$, $M_1^3B_M$, ... are computed, the projection matrix V can be obtained by

$$\text{range}\{V\} = \text{orthogonalize}\{B_M, M_1B_M, M_2B_M, M_1^2B_M, (M_1M_2+M_2M_1)B_M, M_2^2B_M, M_1^3B_M, \ldots \tag{3}$$

Unfortunately, the coefficients quickly become linearly dependent due to numerical instability. In the end, the matrix V is often so inaccurate that it does not possess the expected theoretical properties.

Recursively computing V

The series expansion (2) can also be written into the following formulation:

$$H(s) = \frac{1}{s}[B_M + (\sigma_1M_1 + \sigma_2M_2)B_M + \ldots + (\sigma_1M_1 + \sigma_2M_2)^iB_M + \ldots] \tag{4}$$

Using (4), we define

$$\begin{aligned} R_0 &= B_M, \\ R_1 &= [M_1, M_2]R_0, \\ &\vdots \\ R_j &= [M_1, M_2]R_{j-1}, \\ &\vdots \end{aligned} \tag{5}$$

We see that $R_0, R_1, \ldots, R_j, \ldots$ include all the coefficient matrices in the series expansion (4). Therefore, we can use $R_0, R_1, \ldots, R_j, \ldots$ to generate the projection matrix V:

$$\text{range}\{V\} = \text{colspan}\{R_0, R_1, \ldots, R_m\}. \tag{6}$$

Here, V can be computed employing the recursive relations between R_j, $j = 0, 1, \ldots, m$ combined with the modified Gram-Schmidt process [3].

Improved algorithm for recursively computing V

Note that the coefficients $M_1M_2B_M$ and $M_2M_1B_M$ are two individual terms in (5), which are computed and orthogonalized sequentially within the modified Gram-Schmidt process. Observing that they are actually both coefficients of $\sigma_1\sigma_2$, they can be combined together as one term during the computation as in (3). Based on this, we develop an algorithm which can compute

V in (3) by a modified Gram-Schmidt process. By this algorithm, the matrix V is numerically stable which guarantees the accuracy of the reduced-order model. Furthermore, the size of the reduced-order model is smaller than that of the reduced-order model derived by (6). Therefore, this improved algorithm is optimal for the parametric system considered in this paper.

References

[1] T. Wittig, R. Schuhmann, and T. Weiland. Model order reduction for large systems in computational electromagnetics. *Linear Algebra and its Applications*, 415(2-3):499-530, 2006.

[2] L. Daniel, O.C. Siong, L.S. Chay, K.H. Lee, and J. White. A multiparameter moment-matching model-reduction approach for generating geometrically parameterized interconnect performance models. *IEEE Trans. Comput.-Aided Des. Integr. Circuits Syst.*, 22 (5):678–693, 2004.

[3] L. Feng and P. Benner. A Robust Algorithm for Parametric Model Order Reduction. *Proc. Appl. Math. Mech.*, 7, 2008 (to appear).

Mathematics in Industry and Technology, Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany; benner@mathematik.tu-chemnitz lihong.feng@mathematik.tu-chemnitz.de

This research is supported by the Alexander von Humboldt-Foundation and by the research network *SyreNe — System Reduction for Nanoscale IC Design* within the program *Mathematics for Innovations in Industry and Services* (Mathematik für Innovationen in Industrie und Dienstleistungen) funded by the German Federal Ministry of Education and Science (BMBF).